Shallow Water Equations (SWE): Foundations and Literature Review on Residual Distribution (RD)

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Overview

- Shallow water equations 101
- Derivation of SWE
- SWE components – physical meaning
- SWE literature on RD

WHAT is

Shallow Water Equations (SWE)

- A simplified mathematical model from Navier-Stokes equations which focus on fluid surface profile.

- Originated from 1D Saint-Venant equations

- Common applications of SWE: Modelling of water flow
  - in coastal areas, lakes, estuaries, rivers, reservoirs, open channel flows.
  - To study about bore/tidal wave propagation, wave interaction with bathymetry, stationary hydraulic jump, dam break and flooding, tsunami generation and propagation

WHEN to apply

Shallow Water Equations (SWE)

Physical conditions for SWE:

- Free surface
- “shallow”:

water depth, h << characteristic length of water body, L

- $\frac{h}{L} < 10^{-3} \sim 10^{-4}$
- Incompressible flow: Density is independent of Pressure.

In the case where sediment, salinity and pollution are not considered, density is constant.

Derivation of SWE

- From Navier-Stokes equations (mass and momentum conservation)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{\boldsymbol{u}}) = 0 \qquad (1)$$

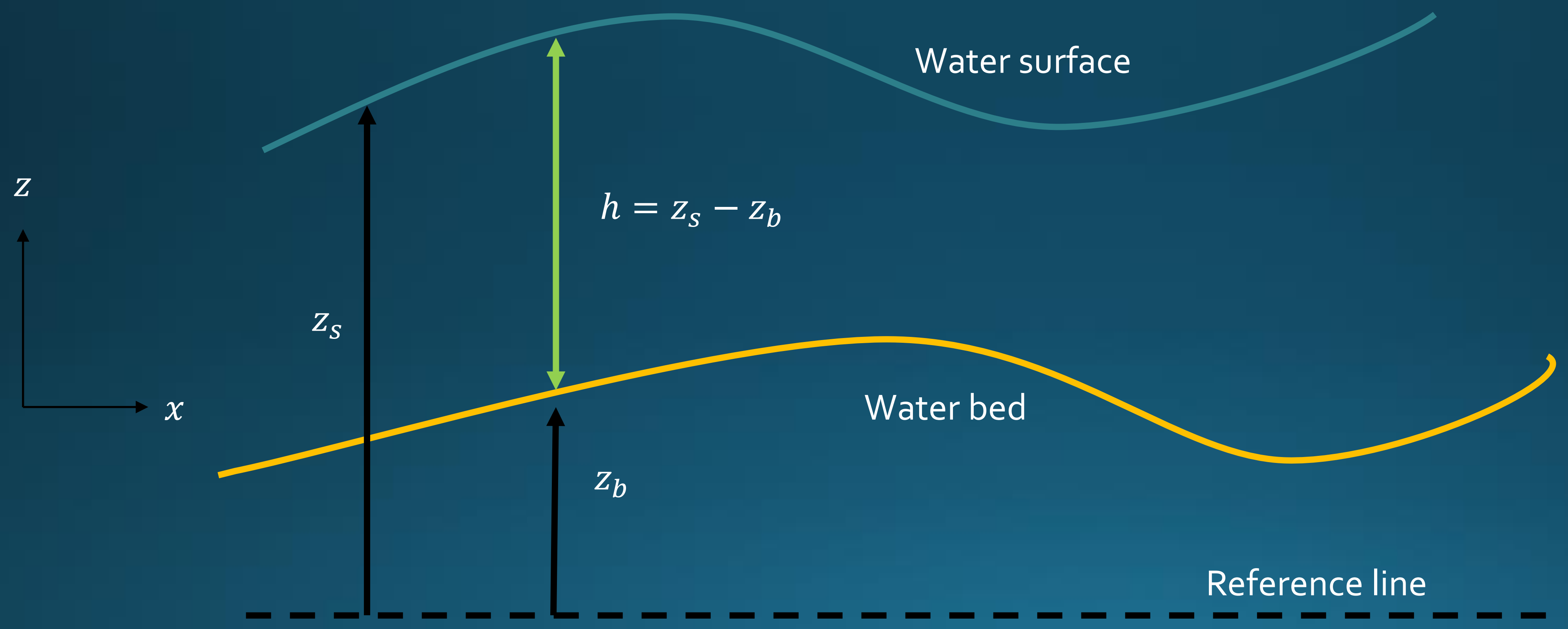
$$\vec{\boldsymbol{u}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\frac{\partial (\rho \vec{\boldsymbol{u}})}{\partial t} + \nabla \cdot (\rho \vec{\boldsymbol{u}} \vec{\boldsymbol{u}}) = \vec{S}_F \qquad (2)$$

- For constant density, equation (1) becomes

$$\nabla \cdot (\vec{\boldsymbol{u}}) = 0 \qquad (3)$$

Derivation of SWE



Boundary condition

- At the bathymetry
  1. No normal flow into the bathymetry

$$w\Big|_{z=z_b} = u\Big|_{z=z_b}\frac{\partial z_b}{\partial x} + v\Big|_{z=z_b}\frac{\partial z_b}{\partial y} \tag{4}$$

- At the water surface
  1. No normal flow out of the water surface

$$w\Big|_{z=z_s} = \frac{\partial z_s}{\partial t} + u\Big|_{z=z_s}\frac{\partial z_s}{\partial x} + v\Big|_{z=z_s}\frac{\partial z_s}{\partial y} \tag{5}$$

Mass Continuity equation

$$\nabla \cdot (\vec{\boldsymbol{u}}) = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (6)$$

Integrating the equation along z from z_b to z_s

$$\int_{z_b}^{z_s} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0 \qquad (7)$$

$$\int_{z_b}^{z_s} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx_3 + \int_{z_b}^{z_s} \left(\frac{\partial w}{\partial z} \right) dz = 0 \qquad (8)$$

Mass Continuity equation

Using Leibniz Integral rule on the first 2 terms

$$\left(\frac{\partial}{\partial x}\int_{z_b}^{z_s} u\,dz + u\Big|_{z=z_b}\frac{\partial z_b}{\partial x} - u\Big|_{z=z_s}\frac{\partial z_s}{\partial x}\right) + \left(\frac{\partial}{\partial y}\int_{z_b}^{z_s} v\,dz + v\Big|_{z=z_b}\frac{\partial z_b}{\partial y} - v\Big|_{z=z_s}\frac{\partial z_s}{\partial y}\right) + \int_{z_b}^{z_s}\left(\frac{\partial w}{\partial z}\right)dz = 0 \quad (9)$$

Collecting boundary terms and integrate the last term,

$$\left(\frac{\partial}{\partial x}\int_{z_b}^{z_s} u\,dz\,\frac{\partial}{\partial y}\int_{z_b}^{z_s} v\,dz +\right) - \left(u\Big|_{z=z_s}\frac{\partial z_s}{\partial x} + v\Big|_{z=z_s}\frac{\partial z_s}{\partial y}\right) + \left(u\Big|_{z=z_b}\frac{\partial z_b}{\partial x} + v\Big|_{z=z_b}\frac{\partial z_b}{\partial y}\right) + w\Big|_{z=z_s} - w\Big|_{z=z_b} = 0 \quad (10)$$

Substitute boundary condition for values of w,

$$\begin{aligned}&\left(\frac{\partial}{\partial x}\int_{z_b}^{z_s} u\,dz + \frac{\partial}{\partial y}\int_{z_b}^{z_s} v\,dz\right) - \left(u\Big|_{z=z_s}\frac{\partial z_s}{\partial x} + v\Big|_{z=z_s}\frac{\partial z_s}{\partial y}\right) + \left(u\Big|_{z=z_b}\frac{\partial z_b}{\partial x} + v\Big|_{z=z_b}\frac{\partial z_b}{\partial y}\right)\\ &+ \left(\frac{\partial z_s}{\partial t} + u\Big|_{z=z_s}\frac{\partial z_s}{\partial x} + v\Big|_{z=z_s}\frac{\partial z_s}{\partial y}\right) - \left(u\Big|_{z=z_b}\frac{\partial z_b}{\partial x} + v\Big|_{z=z_b}\frac{\partial z_b}{\partial y}\right) = 0\end{aligned} \quad (11)$$

Mass Continuity equation

Introducing depth averaging velocities,

$$\bar{u} = \frac{1}{h}\int_{z_b}^{z_s} u\, dz \qquad \bar{v} = \frac{1}{h}\int_{z_b}^{z_s} v\, dz \tag{12a,b}$$

Cancelling boundary terms leaving

$$\frac{\partial z_s}{\partial t} - \frac{\partial z_b}{\partial t} + \left(\frac{\partial}{\partial x}\int_{z_b}^{z_s} u\, dz + \frac{\partial}{\partial y}\int_{z_b}^{z_s} v\, dz + \right) = 0 \tag{13}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) = 0 \tag{14}$$

Momentum Conservation equation(s)

$$\frac{\partial(\rho\vec{\boldsymbol{u}})}{\partial t} + \nabla \cdot (\rho\vec{\boldsymbol{u}}\vec{\boldsymbol{u}}) = \vec{S}_F$$

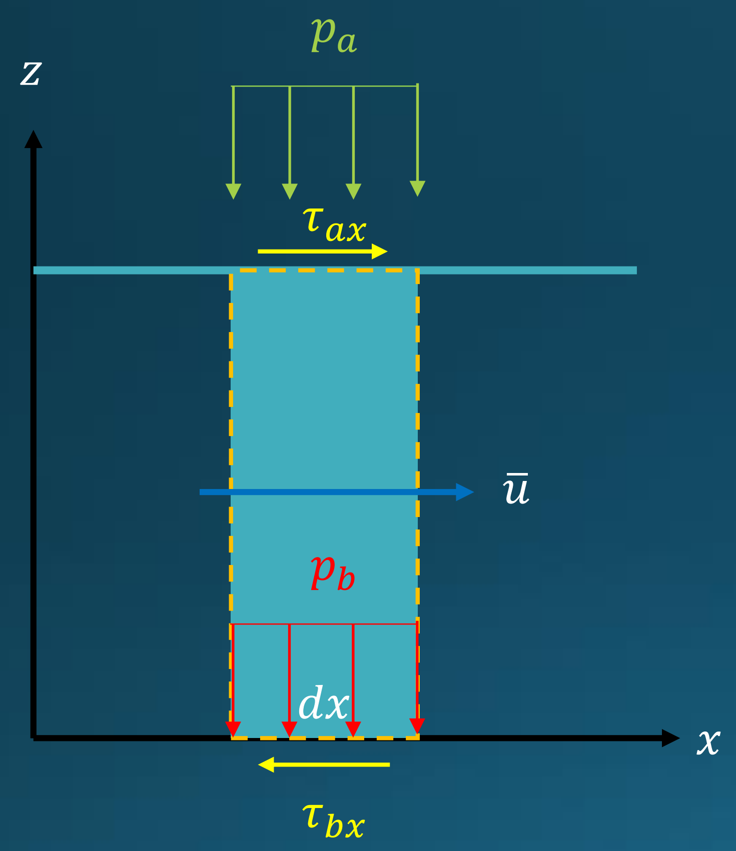
Components for $\vec{S}_F$

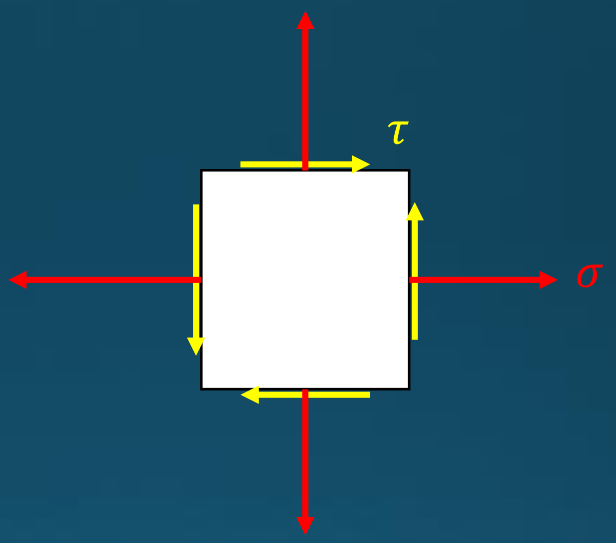
1. Gravity force (chief force) – surface and bottom slope
2. Coriolis inertial force
3. Tide-raising force
4. Frictional force between the flow and bed
5. Wind stress
6. Atmospheric pressure gradient
7. Viscosity

Body Forces (items 2–3)

Momentum Conservation equation(s)

Taking a more direct derivation with more explicit physical meaning





p_a	Atmospheric pressure	p_b	Hydrostatic pressure
τ_{ax}	Wind stress (x-comp)	τ_{bx}	Bottom friction (x-comp)

Momentum Conservation equation(s)

Rate of change of x-momentum of the fluid domain

$$\rho h \frac{D\bar{u}}{Dt} = \rho h \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) \quad (15) \qquad \frac{\partial \bar{u}}{\partial z} = 0 \quad (16)$$

Pressure difference between two lateral faces across dx

$$-h \frac{\partial p_a}{\partial x} - \rho g h \frac{dz_s}{dx} \quad (17)$$

Difference in shear stress on the top and bottom faces

$$\tau_{ax} - \tau_{bx} \quad (18)$$

Difference in shear stress across dy results in viscous forces

Momentum Conservation equation(s)

From Newton's Second Law

$$h\frac{\partial\bar{u}}{\partial t}+h\bar{u}\frac{\partial\bar{u}}{\partial x}+h\bar{v}\frac{\partial\bar{u}}{\partial y}=-\left(\frac{h}{\rho}\frac{\partial p_a}{\partial x}+gh\frac{dz_s}{dx}\right)+\frac{\tau_{ax}-\tau_{bx}}{\rho}+F_B+F_{vis} \qquad (19)$$

Splitting $z_s = \mathrm{h} + z_b$, dropping last 2 terms, changing equation to conservation form

$$\frac{\partial(h\bar{u})}{\partial t}+\frac{\partial(h\bar{u}^2)}{\partial x}+\frac{\partial(h\overline{uv})}{\partial y}=-\left(\frac{h}{\rho}\frac{\partial p_a}{\partial x}+gh\frac{dh}{dx}+gh\frac{dz_b}{dx}\right)+\frac{\tau_{ax}-\tau_{bx}}{\rho} \qquad (20)$$

$$\frac{\partial(h\bar{u})}{\partial t}+\frac{\partial\left(h\bar{u}^2+\frac{1}{2}gh^2\right)}{\partial x}+\frac{\partial(h\overline{uv})}{\partial y}=-\left(\frac{h}{\rho}\frac{\partial p_a}{\partial x}+gh\frac{dz_b}{dx}\right)+\frac{\tau_{ax}-\tau_{bx}}{\rho} \qquad (21)$$

Shallow Water Equations - Components

$$\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{F}(U) + \vec{S}(U) = 0 \qquad (22)$$

For 2D case

$$\vec{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} \quad \vec{F} = [F_X \quad F_Y] = \begin{bmatrix} hu & hv \\ hu^2 + \frac{gh^2}{2} & huv \\ huv & hv^2 + \frac{gh^2}{2} \end{bmatrix} \quad \vec{S} = \begin{bmatrix} 0 \\ gh\frac{\partial b(x,y)}{\partial x} + \frac{h}{\rho}\frac{\partial p_a}{\partial x} - \frac{\tau_{ax}}{\rho} + \frac{\tau_{bx}}{\rho} \\ gh\frac{\partial b(x,y)}{\partial y} + \frac{h}{\rho}\frac{\partial p_a}{\partial y} - \frac{\tau_{ay}}{\rho} + \frac{\tau_{bx}}{\rho} \end{bmatrix}$$

Pressure gradient due to variation of depth

Bottom topography

Atmospheric pressure gradient

Wind Stress

Bottom Friction

Shallow Water Equations - Components

1. Atmospheric Pressure Gradient
   - Dominant in storm-surge forecast
   - p_a will be modelled as cyclone in terms of r, radius from center of typhoon

2. Surface Wind stress
   - Estimated by a semi-theoretical formula, based on similarity hypothesis proposed by Karman
   - $\tau_{a,i} = \rho_a \left\{ \frac{\kappa u_{w,i}}{ln[(z+k)/k]} \right\}^2$

3. Bottom Friction
   - $\frac{\tau_{bx}}{\rho}$ can be expressed as $\frac{n^2 ghu^2}{R^{4/3}}$ in hydraulics approach
   - n is the Manning roughness coefficient

Shallow Water Equations - Components

Body Forces

1. Coriolis inertial force
   - $F_{C,x} = fv,\ F_{C,y} = -fu$
   - $f = 2\omega sin\varphi$

2. Tide- raising force
   - Newton's universal gravitation exerted on a fluid body and mainly from the moon and the sun.

SWE RD solver – Literature Review

- Prior to year 2000, most SWE solvers are based on Finite Volume method.
- M. Ricchiuto, R. Abgrall, H. Deconinck(2007) apply RD method on SWE achieving second order accuracy for steady cases
- M. Ricchiuto, A. Bollermann (2009) obtained second order accuracy results with positivity in dry areas
- D.Sarmany, M.E. Hubbard (2013) achieved similar performance in rotational flow
- D. Sarmany, M. E. Hubbard, M. Ricchiuto (2013) used blended discontinuous in time scheme, still, results are close to second order accuracy.

SWE RD solver – Literature Review

- M. Ricchiuto (2015) reported a more efficient version of SWE RD solver (explicit Lax-Friedrichs) with second order accuracy, however, cases with irregular bathymetry reduces the order of accuracy to first order and is 'solved' by using structured grid or grid adaptation technique.
- There is no SWE RD solver reported that apply the second law of thermodynamics to ensure entropy consistency in literature.

SWE solver with other methods

- F. Bouchut, T. Morales de Luna (2008) applied entropy satisfying scheme on SWE but on a FV frame (without eigenvalues)
- U.S. Fjordholm, S. Mishra, E. Tadmor (2011) introduced an energy stable scheme on SWE, again on FV frame
- N-J. Wu, C. Chen, T-K. Tsay (2016) proposed a weighted-least-square local polynomial approximation to 2D SWE problems, it is a meshless scheme but without entropy consistency
- N. Izem, M. Seaid, M. Wakrim (2016) studied on Discontinuous Galerkin Method on SWE with FV and FE to produce high order accuracy

RD solver on other equations

- M. Ricchiuto, R. Abgrall (2010) obtained second order accuracy with unsteady flow on hyperbolic-diffusion equations with mass lumping technique
- A. Mazaheri, H.Nishikawa (2015, 2016) are working in the similar equations and Burgers' equations and claimed to be able to capture shock more accurately by introducing characteristic-based nonlinear wave sensor

Thank you