Shallow Water Equations Ringleb's Flow

By Chang Wei Shyang

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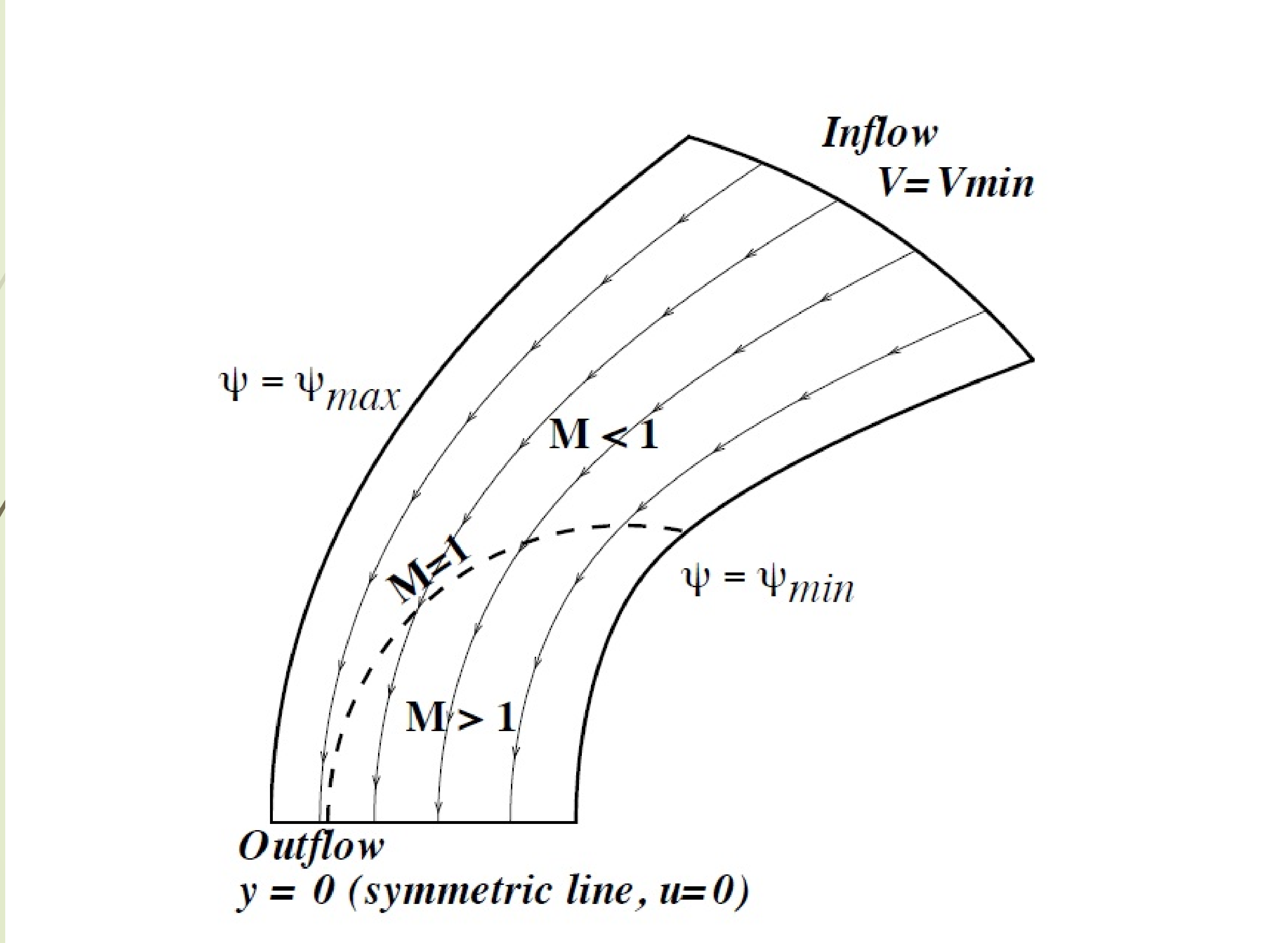
USM Aerospace Engineering CFD Research Group

Ringleb's flow

- A particular flow where the fluid accelerates from subsonic to supersonic regime smoothly without any shock.
- Much useful to test the accuracy of an inviscid flow for both subsonic and supersonic flow as the exact solution is available.
- Limited to gas flow with $\gamma = 1.4$, governed by Euler equations
- A good candidate for order of accuracy test on shallow water equations.

Ringleb's flow



Ringleb Flow: A flow in a channel, being smoothly accelerated from subsonic to supersonic. The supersonic region can be adjusted by varying ψ

Ringleb's flow – Euler derivation with SWE analogy

- Start with potential and stream equations,

$$u = \frac{\partial \phi}{\partial x}$$

$$v = \frac{\partial \phi}{\partial y}$$

$$u = \frac{\rho_0}{\rho}\frac{\partial \psi}{\partial y}$$

$$v = -\frac{\rho_0}{\rho}\frac{\partial \psi}{\partial x}$$

SWE

$$u = \frac{h_0}{h}\frac{\partial \psi}{\partial y}$$

$$v = -\frac{h_0}{h}\frac{\partial \psi}{\partial x}$$

- Introduce flow quantities, V and θ

$$u = V cos\theta$$

$$v = V sin\theta$$

Ringleb's flow – Euler derivation with SWE analogy

$$d\phi = \phi_x dx + \phi_y dy = V(cos\theta dx + sin\theta dy)$$

$$d\psi = \psi_x dx + \psi_y dy = \left(\frac{\rho}{\rho_0}\right) V(-sin\theta dx + cos\theta dy)$$

$$dx = \frac{cos\theta}{V} d\phi - \frac{\rho_0}{\rho}\frac{sin\theta}{V} d\psi$$

$$dy = \frac{sin\theta}{V} d\phi + \frac{\rho_0}{\rho}\frac{cos\theta}{V} d\psi$$

$$V(x,y), \theta(x,y)$$

$$\phi(x,y), \psi(x,y)$$

Ringleb's flow – Euler derivation with SWE analogy

$$d\phi = \frac{\partial \phi}{\partial V} dV + \frac{\partial \phi}{\partial \theta} d\theta = \phi_V dV + \phi_\theta d\theta$$

$$d\psi = \frac{\partial \psi}{\partial V} dV + \frac{\partial \psi}{\partial \theta} d\theta = \psi_V dV + \psi_\theta d\theta$$

$$dx = (\frac{cos\theta}{V}\phi_V - \frac{\rho_0}{\rho}\frac{sin\theta}{V}\psi_V)dV + (\frac{cos\theta}{V}\phi_\theta - \frac{\rho_0}{\rho}\frac{sin\theta}{V}\psi_\theta)d\theta$$

$$dy = (\frac{sin\theta}{V}\phi_V + \frac{\rho_0}{\rho}\frac{cos\theta}{V}\psi_V)dV + (\frac{sin\theta}{V}\phi_\theta + \frac{\rho_0}{\rho}\frac{cos\theta}{V}\psi_\theta)d\theta$$

Ringleb's flow – Euler derivation with SWE analogy

V and θ are independent of each other.

$$\frac{\partial x}{\partial V} = x_V = \frac{cos\theta}{V}\phi_V - \frac{\rho_0}{\rho}\frac{sin\theta}{V}\psi_V$$

$$\frac{\partial x}{\partial \theta} = x_\theta = \frac{cos\theta}{V}\phi_\theta - \frac{\rho_0}{\rho}\frac{sin\theta}{V}\psi_\theta$$

$$\frac{\partial y}{\partial V} = y_V = \frac{sin\theta}{V}\phi_V + \frac{\rho_0}{\rho}\frac{cos\theta}{V}\psi_V$$

$$\frac{\partial y}{\partial \theta} = y_\theta = \frac{sin\theta}{V}\phi_\theta + \frac{\rho_0}{\rho}\frac{cos\theta}{V}\psi_\theta$$

Ringleb's flow – Euler derivation with SWE analogy

$$\frac{\partial^2 x}{\partial V \partial \theta} = \frac{\partial^2 x}{\partial \theta \partial V}$$

$$\frac{\partial^2 y}{\partial \theta \partial V} = \frac{\partial^2 y}{\partial V \partial \theta}$$

$$-\frac{sin\theta}{V}\phi_V - \frac{\rho_0}{\rho}\frac{cos\theta}{V}\psi_V = -\frac{cos\theta}{V^2}\phi_\theta + \frac{\rho_0}{\rho}\frac{sin\theta}{V^2}\psi_\theta - \frac{sin\theta}{V}\left[\frac{d}{dV}\left(\frac{\rho_0}{\rho}\right)\right]\psi_\theta$$

$$\frac{cos\theta}{V}\phi_V - \frac{\rho_0}{\rho}\frac{sin\theta}{V}\psi_V = -\frac{sin\theta}{V^2}\phi_\theta - \frac{\rho_0}{\rho}\frac{cos\theta}{V^2}\psi_\theta + \frac{cos\theta}{V}\left[\frac{d}{dV}\left(\frac{\rho_0}{\rho}\right)\right]\psi_\theta$$

Ringleb's flow – Euler derivation with SWE analogy

multiply by $cos\theta$, $sin\theta$ then sum,

$$\phi_\theta = \left(\frac{\rho_0}{\rho}\right) V \psi_V$$

multiply by $sin\theta$, $cos\theta$ then subtract,

$$\phi_V = \left[-\frac{\rho_0}{\rho}\frac{1}{V} + \frac{d}{dV}\left(\frac{\rho_0}{\rho}\right)\right]\psi_\theta = V\left[\frac{d}{dV}\left(\frac{1}{V}\frac{\rho_0}{\rho}\right)\right]\psi_\theta$$

Ringleb's flow – Euler derivation with SWE analogy

For isentropic flow, $\frac{dp}{d\rho} = c^2$ (speed of sound)

Euler's equation for frictionless, irrotational motion,

$$dp = -\rho V\,dV$$

$$\frac{d}{dV}\left(\frac{\rho_0}{\rho}\right) = -\frac{\rho_0}{\rho^2}\frac{d\rho}{dV} = -\frac{\rho_0}{\rho^2}\frac{d\rho}{dp}\frac{dp}{dV}$$

$$= -\frac{\rho_0}{\rho^2}\left(\frac{1}{c^2}\right)(-\rho V) = -\frac{\rho_0}{\rho^2}\frac{V}{c^2}$$

Ringleb's flow – Euler derivation with SWE analogy

SWE

Bernoulli equation

$$\frac{V^2}{2} + gz + \frac{p}{\rho} = constant$$

$$\frac{V^2}{2} + gh = constant$$

$$gh = k - \frac{V^2}{2}$$

$$h = k - \frac{V^2}{2g}$$

$$\frac{dh}{dV} = -\frac{1}{g}V$$

$$\frac{d}{dV}\left(\frac{h_0}{h}\right) = -\frac{h_0}{h^2}\frac{dh}{dV}$$

$$= \frac{h_0}{h^2}\left(-\frac{V}{g}\right) = \frac{h_0}{h}\frac{V}{gh}$$

$$c = \sqrt{gh}$$

Ringleb's flow – Euler derivation with SWE analogy

$$\phi_V = -\frac{\rho_0}{\rho}\frac{1}{V}\left(1-\frac{V^2}{c^2}\right)\psi_\theta$$

$$\phi_{\theta V} = \phi_{V\theta}$$

$$V^2\psi_{VV} + V\left(1+\frac{V^2}{c^2}\right)\psi_V + \left(1-\frac{V^2}{c^2}\right)\psi_{\theta\theta} = 0$$

$$\boxed{c^2 = c_0^2 - \frac{k-1}{2}V^2}$$

$$V^2\left(1-\frac{k-1}{2}\frac{V^2}{c_0^2}\right)\psi_{VV} + V\left(1-\frac{k-3}{2}\frac{V^2}{c_0^2}\right)\psi_V + \left(1-\frac{k+1}{2}\frac{V^2}{c_0^2}\right)\psi_{\theta\theta} = 0$$

Ringleb's flow – Euler derivation with SWE analogy

SWE

Bernoulli equation

$$\frac{V^2}{2} + gz + \frac{p}{\rho} = constant$$

$$\frac{V^2}{2} + gh = constant$$

$$gh = k - \frac{V^2}{2}$$

$$gh_0 = k - \frac{V_0^2}{2}$$

$$k = gh_0$$

$$gh = gh_0 - \frac{V^2}{2}$$

$$c^2 = c_0^2 - \frac{V_0^2}{2}$$

Ringleb's flow – Euler derivation with SWE analogy

$$\psi = P(V) \cdot Q(\theta)$$

$$\psi_V = QP'; \psi_{VV} = QP"; \psi_{\theta\theta} = PQ"$$

$$V^2 \left[\frac{1 - \frac{k-1}{2}\frac{V^2}{c_0^2}}{1 - \frac{k+1}{2}\frac{V^2}{c_0^2}} \right] \frac{P"}{P} + V \left[\frac{1 - \frac{k-3}{2}\frac{V^2}{c_0^2}}{1 - \frac{k+1}{2}\frac{V^2}{c_0^2}} \right] \frac{P'}{P} + \frac{Q"}{Q} = 0$$

V and θ independent ;

$\frac{Q"}{Q} = -n^2$ (simple harmonic equation)

n=1,$P = \frac{1}{V}$

Ringleb's flow – Euler derivation with SWE analogy

Hence, let

$$\psi = \frac{c_0^2 l}{V} sin\theta$$

$$\psi_V = -\frac{c_0^2 l}{V^2} sin\theta$$

$$\psi_\theta = \frac{c_0^2 l}{V} cos\theta$$

$$\phi_V = -c_0^2 l \frac{\rho_0}{\rho} \frac{1}{V^2} \left(1 - \frac{V^2}{c^2}\right) cos\theta$$

$$\phi_\theta = -c_0^2 l \frac{\rho_0}{\rho} \frac{1}{V} sin\theta$$

$$x_V = c_0^2 l \frac{\rho_0}{\rho} \left(-\frac{cos2\theta}{V^2} + \frac{cos^2\theta}{Vc^2}\right)$$

$$x_\theta = -c_0^2 l \frac{\rho_0}{\rho} \frac{sin2\theta}{V^2}$$

$$y_V = c_0^2 l \frac{\rho_0}{\rho} \left(-\frac{sin2\theta}{V^2} + \frac{sin\theta cos\theta}{Vc^2}\right)$$

$$y_\theta = c_0^2 l \frac{\rho_0}{\rho} \frac{cos2\theta}{V^2}$$

Ringleb's flow – Euler derivation with SWE analogy

$$x = \int x_\theta d\theta + f(V) = c_0^2 l \frac{\rho_0}{\rho} \frac{cos 2\theta}{V^2} + f(V)$$

$$x_V = \frac{c_0^2 l}{2} cos 2\theta \left[-\frac{\rho_0}{\rho} \frac{2}{V^3} + \frac{1}{V^2} \left(\frac{\rho_0}{\rho} \frac{V}{c^2} \right) \right] + f'(V)$$

$$f'(V) = \frac{c_0^2}{2} \frac{\rho_0}{\rho} \frac{1}{V c^2}$$

$$f(V) = \frac{c_0^2}{2} \int \frac{\rho_0}{\rho} \frac{dV}{V c^2} = \frac{l}{2} L$$

$$L = -\left[\frac{1}{2} \ln \frac{1+\gamma}{1-\gamma} - \frac{1}{\gamma} - \frac{1}{3\gamma^2} - \frac{1}{5\gamma^5} \right]$$

Ringleb's flow – Euler derivation with SWE analogy

SWE

$$f'(V) = \frac{c_0^2}{2}\frac{h_0}{h}\frac{1}{Vc^2}$$

$$f(V) = \frac{c_0^2}{2}\int \frac{h_0}{h}\frac{dV}{Vc^2}$$

$$= \frac{1}{2}\int \frac{h_0}{h}\frac{gh_0}{gh}\frac{dV}{V} = \frac{l}{2}L$$

$$L = \frac{1}{2}\left[\frac{1}{b} + \ln\frac{1-b}{b}\right]$$

$$b = \frac{h}{h_0} = 1 - \frac{V^2}{2c_0^2}$$

Ringleb's flow – Euler derivation with SWE analogy

$$\frac{x}{l} = \frac{1}{2}\left[\frac{\rho_0}{\rho}\left(\frac{c_0}{V}\right)^2 cos\theta + L\right]$$

$$\frac{y}{l} = \frac{1}{2}\frac{\rho_0}{\rho}\left(\frac{c_0}{V}\right)^2 sin2\theta$$

$$\frac{x}{l} = \frac{1}{2}\frac{\rho_0}{\rho}\left(\frac{c_0^2}{V^2} - 2\frac{\psi^2}{l^2 c_0^2}\right) + \frac{L}{2}$$

$$\frac{y}{l} = \pm\frac{\rho_0}{\rho}\frac{c_0}{V}\frac{\psi}{c_0 l}\sqrt{1 - \frac{V^2}{c_0^2}\left(\frac{\psi}{l c_0}\right)^2}$$

Equations to build up Ringleb's flow domain

Ringleb's flow – Euler derivation with SWE analogy

- The following equations are required to be implemented in Euler code to compute positions of boundary and exact solution.

$$\psi = \frac{\sin\theta}{V},$$

$$x(\psi, V) = \frac{1}{\rho}\left[\frac{1}{2V^2} - \psi^2\right] + \frac{L}{2},$$

$$y(\psi, V) = \pm\frac{\psi}{\rho V}\sqrt{1 - V^2\psi^2},$$

$$\rho = b^5,$$

$$L = \frac{1}{b} + \frac{1}{3b^3} + \frac{1}{5b^5} - \frac{1}{2}\ln\left(\frac{1+b}{1-b}\right),$$

$$b = \sqrt{1 - 0.2V^2},$$

$$p = b^7.$$

Ringleb's flow – Euler derivation with SWE analogy

$$\rho = b^5,$$

$$L = \frac{1}{b} + \frac{1}{3b^3} + \frac{1}{5b^5} - \frac{1}{2}\ln\left(\frac{1+b}{1-b}\right),$$

$$b = \sqrt{1 - 0.2V^2},$$

SWE

$$h = b$$

$$L = \frac{1}{2}\left[\frac{1}{b} + \ln\frac{1-b}{b}\right]$$

$$b == 1 - \frac{V^2}{2}$$

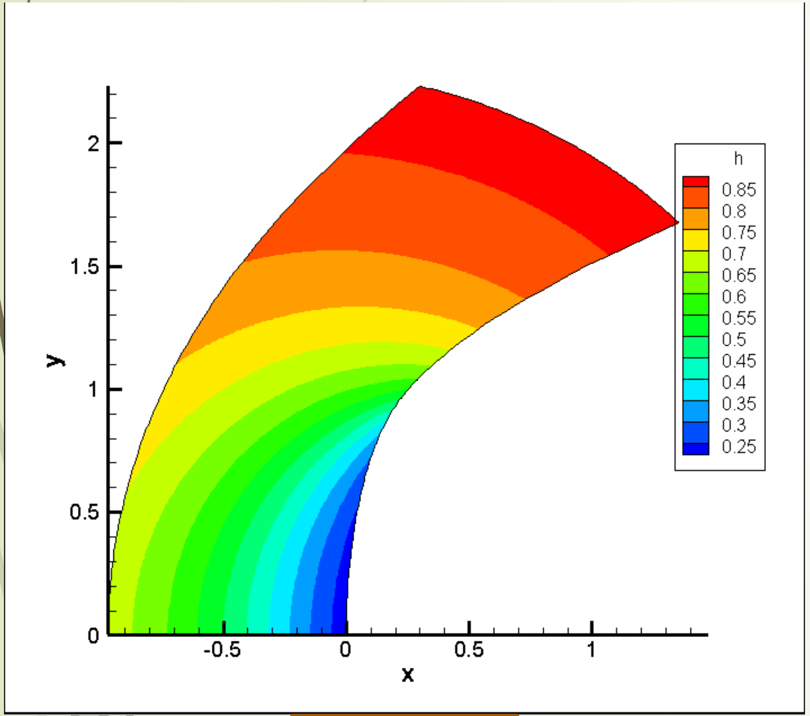
Variables have been non-dimensionalized by stagnation values

Ringleb's flow

- Preliminary results

Ringleb's flow – Preliminary Results

Exact Solutions



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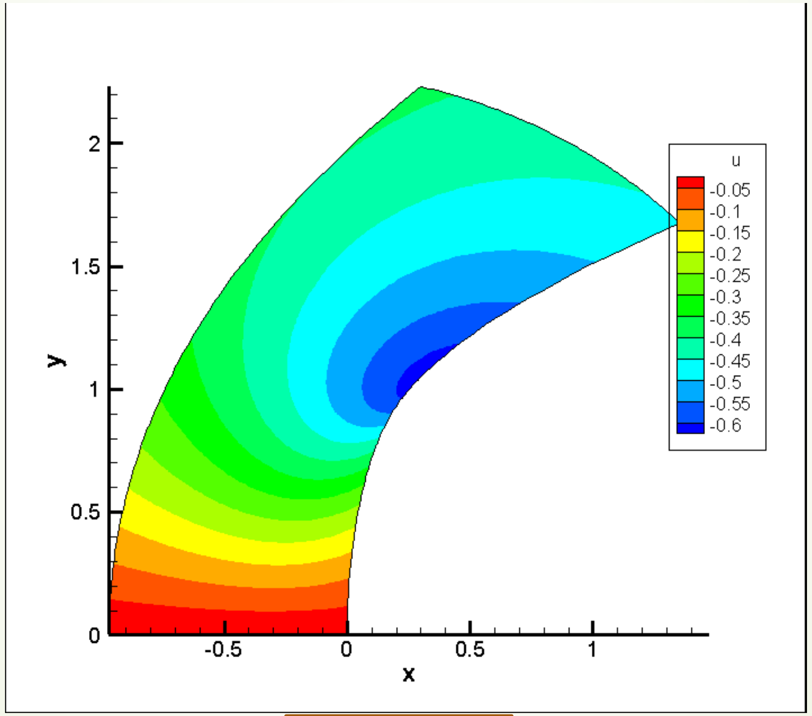

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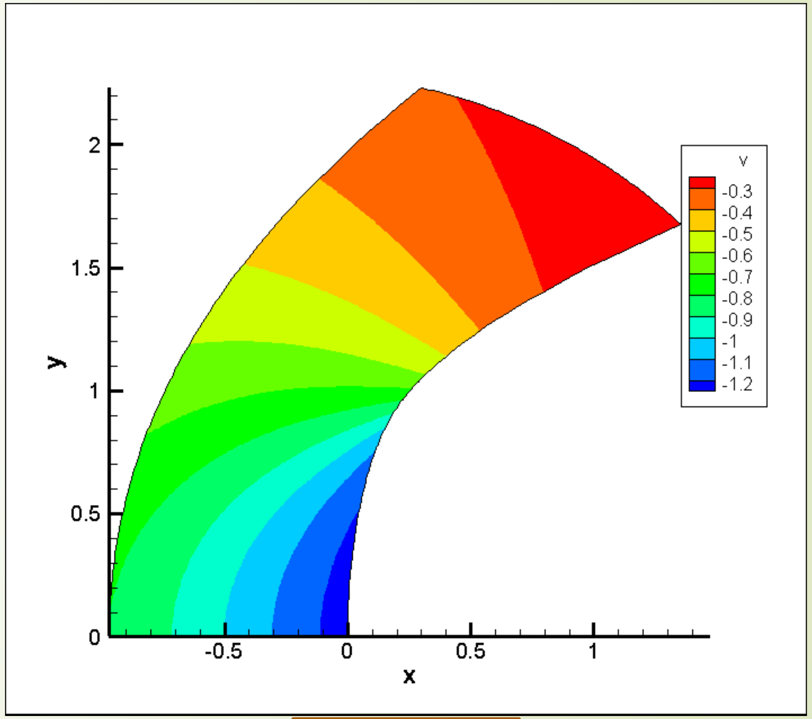
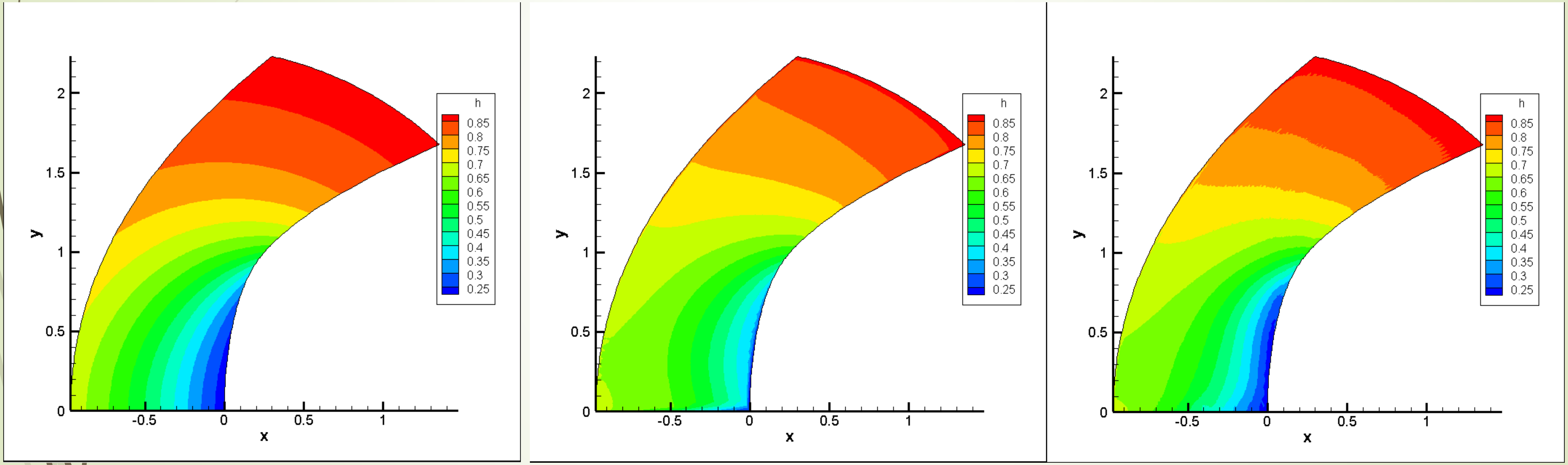


v

Ringleb's flow – Preliminary Results

H (81 X 81 grid)



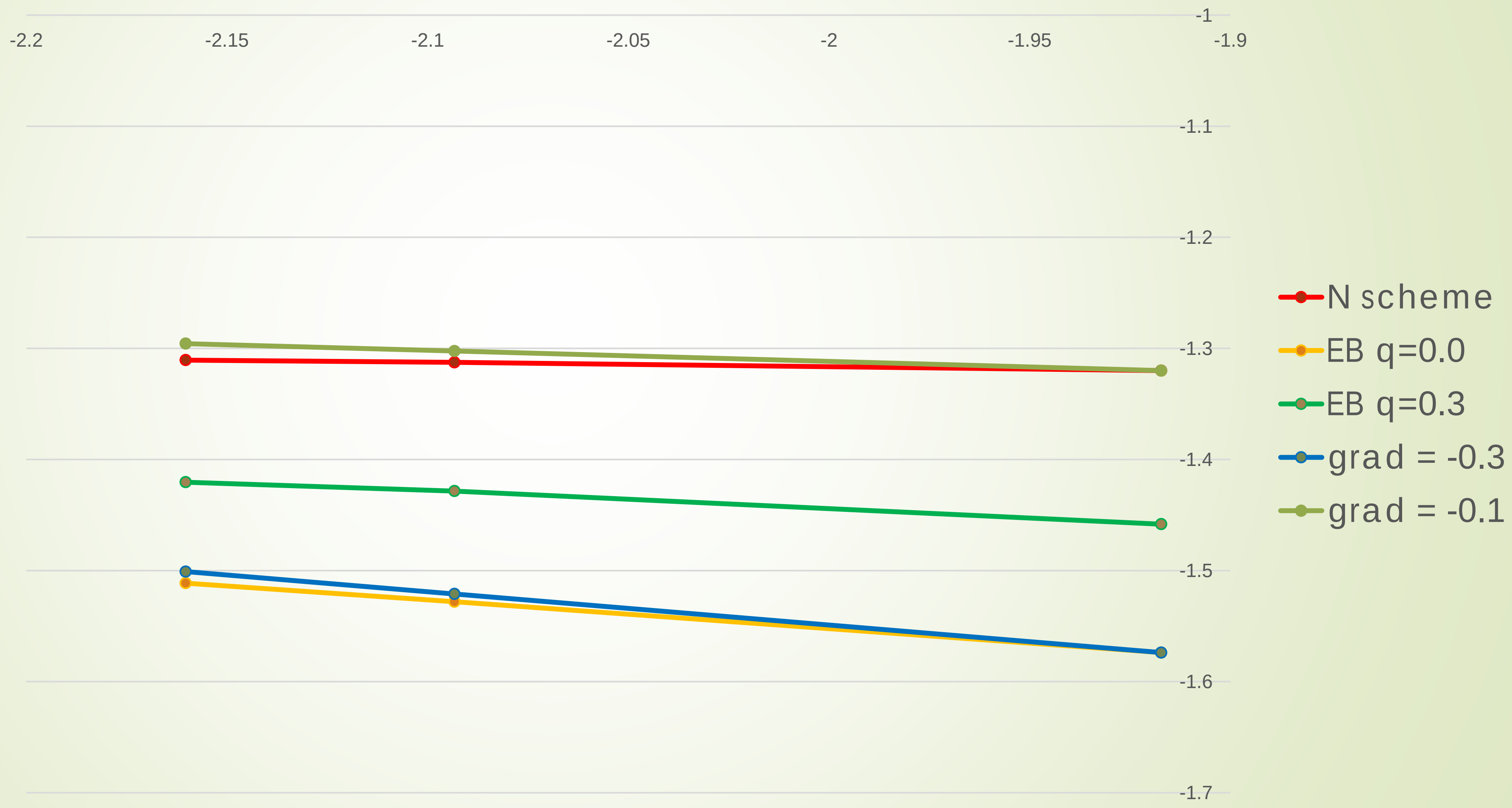
Exact Solutions

N scheme

Entropy baseline
q = 0

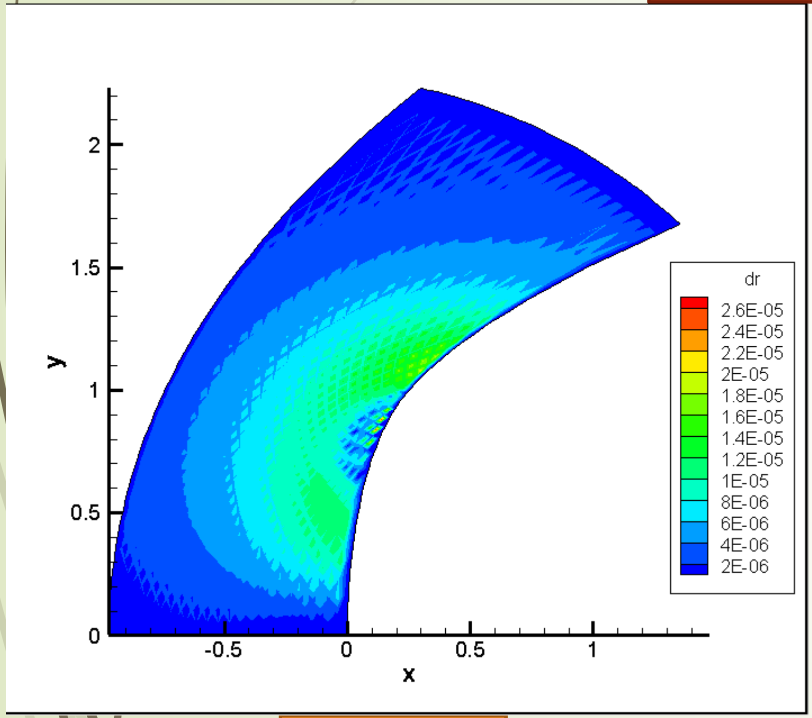
Ringleb's flow – Preliminary Results

Negative order of accuracy!



Ringleb's flow – Preliminary Results

Error Contours at 1st iteration – N scheme



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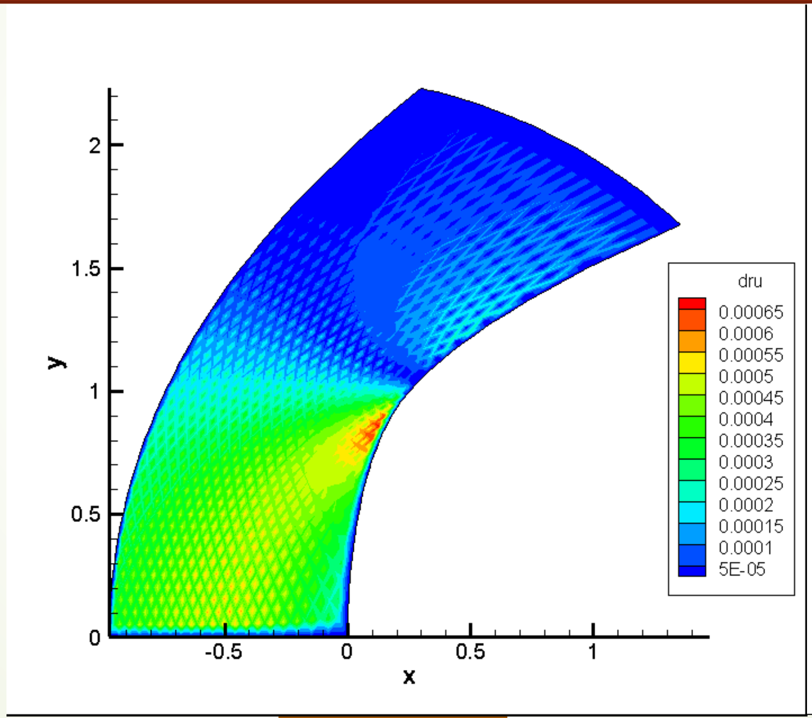

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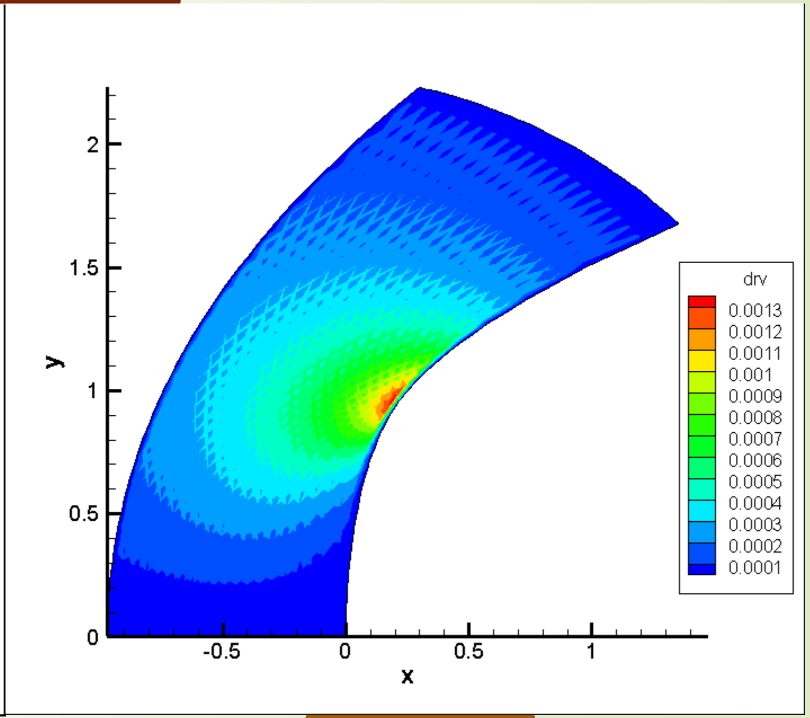

hv

Current proposals to resolve negative order of accuracy

- Double check the model with FV approach
- Inaccurate presentation of flow physics with Bernoulli equations?
- Incorporate hydraulic energy equation to SWE to preserve flow physics