Energy-Stable Residual Distribution Methods for the System of 2D Shallow Water Equations

CFD Group Presentation

Chang Wei Shyang

wshyang88@gmail.com
School of Aerospace Engineering

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Overview

- RD from Scalar to System of Equations
  - Recap of Flux Difference RD for Scalar Equation
  - Entropy Control as the Additional Constraint
- Shallow Water Equations (SWE)
  - Governing Equations
  - Entropy Function
  - Well-balancedness property
- Energy-Stable RD Methods
  - Energy Conservation
  - Energy Stability
  - First Order Method
  - Second Order Method
  - Well-balanced ESRD
- Results

Given the scalar equation,

$$u_t + \vec{\nabla} \cdot \vec{f}(u) = 0. \tag{1}$$

Considering steady problem, the element residual of the Flux Difference RD method is defined as,

$$\phi^{\mathbb{E}} = -\iint \vec{\nabla} \cdot \vec{f}(u) dA. \tag{2}$$

which can be further discretized by trapezoidal rule to be

$$\phi^{\mathbb{E}} = \frac{1}{2} \sum_{p} (\vec{f}_p - \vec{f}^*) \cdot \vec{n}_p, \quad p = i, j, k. \tag{3}$$

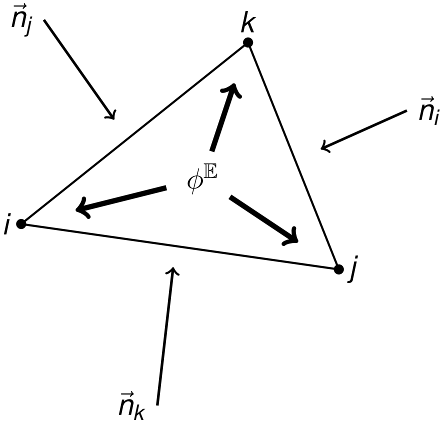


Figure: Residual distributed over triangular element.

The distribution of the signals to node i via Flux Difference RD is

$$\phi_i = \underbrace{\frac{1}{2}(\vec{f}_i - \vec{f}^*) \cdot \vec{n}_i}_{\phi_i^{\text{iso}}} \underbrace{-\alpha[u]_{ji} - \beta[u]_{kj} - \gamma[u]_{ik}}_{\phi_i^{\text{art}}} . \tag{4}$$

where $\vec{f}^*$, α, β, γ are degrees of freedom of this scheme which can be employed to recover classic RD methods or to fulfil certain additional constraints.

Isotropic signals

- Using **arithmetic average of three nodal values** of the variable for $\vec{f}^* = \vec{f}(\bar{u})$ within the element is able to achieve **third order accuracy** for simple linear advection problems.
- However, it is a **central** approach and not stable for hyperbolic problems.

Artificial signals

- Serves as a form of artificial diffusion to attain numerical **stability**.
- Specific choices of α, β, γ can recover classic RD methods or to achieve condition for positivity.

The weak solutions of the conservation laws of mass, momentum and energy are **not necessarily unique** and thus require additional constraint(s). One good choice of this criterion is the **Second Law of Thermodynamics.** Based on this law, the following properties are used.

- **Entropy Conservation**: The entropy variable is conserved with zero entropy production.
- **Entropy Stability**: The entropy generation is captured with the correct sign.
- **Entropy Consistency**: The entropy generation is captured with the correct sign and exact physical amount.

The degrees of freedom aforementioned are designed to fulfil different entropy properties.

- **Isotropic signals** ($\vec{f}^*$): Entropy Conservation
- **Artificial signals** (α, β, γ): Entropy Stability

For scalar equation, the shock-tree or expansion problem adhering to the Burgers' equation are tested with positive results.

This entropy control algorithm was extended to systems of equations such as Euler Equations and Shallow Water Equations.

- A *simplified* mathematical model from Navier-Stokes Equations which focus on *fluid surface profile*.
- SWE is used to model *open channel flows*. E.g. coastal areas, lakes, estuaries, rivers, reservoirs, etc.
- *Common applications* of interest: Bore/tidal wave propagation, wave interaction with bottom topography, stationary hydraulic jump, dam break and flooding, tsunami generation and propagation.

Physical conditions for SWE: (When to use it?)

- Free surface
- "shallow", meaning that the water depth is very small in comparison with the characteristic length of the water body.

$$\frac{h}{L} < 10^{-3} \sim 10^{-4}$$

- Vertical velocity are negligible.
- Variation of horizontal velocities across the depth is small. Thus, they can be represented by averaged values.
- The fluid is incompressible.

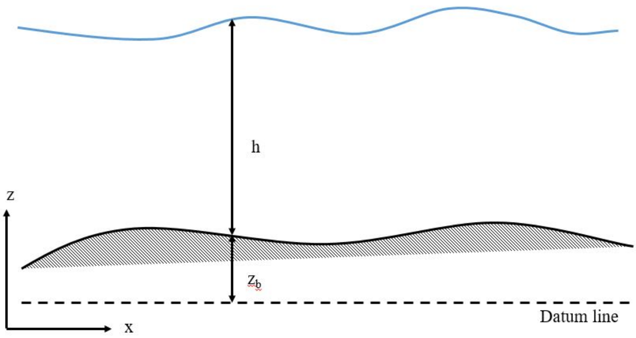


Figure: Some denotations for SWE.

The compact form of SWE is

$$\boldsymbol{u}_t + \vec{\nabla} \cdot \vec{\boldsymbol{f}}(\boldsymbol{u}) + \boldsymbol{s}(\boldsymbol{u}) = 0. \tag{5}$$

where

$$\boldsymbol{u} = \begin{bmatrix} h & hu & hv \end{bmatrix}^T, \tag{6}$$

and for 2D, the fluxes are

$$\vec{\boldsymbol{f}}(\boldsymbol{u}) = \begin{bmatrix} hu & hv \\ hu^2 + \frac{1}{2}gh^2 & huv \\ huv & hv^2 + \frac{1}{2}gh^2 \end{bmatrix}. \tag{7}$$

and $\boldsymbol{s}(\boldsymbol{u})$ is the source terms which includes the contributions from various types of mass or momentum sources.

Examples of source terms

- **Bottom topography/bathymetry** - momentum
- Rain - mass
- Infiltration rate - mass
- Fluid viscosity - momentum
- Wind Stress - momentum
- Bottom friction - momentum
- Atmospheric pressure gradient - momentum
- Coriolis force - momentum

With only the bottom topography,

$$\boldsymbol{s}(\boldsymbol{u}) = \begin{bmatrix} 0 & gh\frac{\partial z_b}{\partial x} & gh\frac{\partial z_b}{\partial y} \end{bmatrix}^T. \tag{8}$$

There is a convex entropy function for SWE which also coincides with the height-averaged mechanical energy,

$$E = \frac{hu^2 + hv^2 + gh^2}{2} + ghz_b, \tag{9}$$

which satisfies the energy dissipation inequality

$$\frac{\partial E}{\partial t} + \frac{\partial (F + S_1)}{\partial x} + \frac{\partial (G + S_2)}{\partial y} \leq 0, \tag{10}$$

where

$$F = (hu^3/2 + gh^2 u + huv^2/2),$$

$$G = (hv^3/2 + gh^2 v + hu^2 v/2),$$

$$S_1 = ghz_b u, \quad S_2 = ghz_b v.$$

Connecting the entropy function to the governing equations...
Introducing symmetrizing / energy variables,

$$\boldsymbol{v} = \frac{\partial E}{\partial \boldsymbol{u}} = (-(u^2 + v^2)/2 + g(h + z_b), u, v)^T, \tag{11}$$

Next, direct computation shows that

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{f} &= F + \frac{gh^2 u}{2} + ghz_b u, \\ \boldsymbol{v} \cdot \boldsymbol{g} &= G + \frac{gh^2 v}{2} + ghz_b v. \end{aligned} \tag{12}$$

Well-balancedness

- A numerical property closely related to the discretization of the source term.
- The **numerical balance** between the flux divergence and the source term modelling of the bottom topography.
- The ability of a numerical scheme to preserve the steady **lake at rest** case.

<u>Well-balancedness</u>

Considering nodal values at point i with the initial conditions of

$$u_i \equiv v_i \equiv 0, \quad h_i + z_{b,i} \equiv \text{constant} \quad \forall i, \tag{13}$$

the numerical solution has to satisfies

$$\frac{d}{dt} h_i \equiv 0, \quad \frac{d}{dt} h_i(u_i, v_i) \equiv 0 \quad \forall i. \tag{14}$$

Energy-Stable Residual Distribution Method (ESRD)

Following the works of [Ismail and Chizari, 2017] for scalar equations, the flux difference residual distribution method is recalled in systems form,

$$\phi_p = \phi_p^{\text{iso}} + \phi_p^{\text{art}}. \tag{15}$$

The same concept from the scalar version is utilized to achieve energy control.

- **Isotropic signals** : Energy Conservation
- **Artificial signals** : Energy Stability

The isotropic signals for each node are designed to have the following form,

$$\phi_p^{\text{iso}} = \frac{1}{2}((\boldsymbol{f}_p, \boldsymbol{g}_p) - (\boldsymbol{f}^*, \boldsymbol{g}^*)) \cdot \vec{n}_p \tag{16}$$

From Eq. (10), energy conservation requires

$$\dot{E} = \sum_p (F_p, G_p) \cdot \vec{n}_p - \sum_p (\boldsymbol{v}_p^T \phi_p) = 0. \tag{17}$$

After series of mathematical operations, $(\vec{\boldsymbol{f}}^{*})$ is determined to be

$$\boldsymbol{f}^{C} = \begin{bmatrix} \bar{h}\bar{u} \\ \bar{h}\bar{u}^{2} + \frac{1}{2}g\bar{h}^{2} \\ \bar{h}\bar{u}\bar{v} \end{bmatrix}, \quad \boldsymbol{g}^{C} = \begin{bmatrix} \bar{h}\bar{v} \\ \bar{h}\bar{u}\bar{v} \\ \bar{h}\bar{v}^{2} + \frac{1}{2}g\bar{h}^{2} \end{bmatrix} \tag{18}$$

and deemed $(\vec{\boldsymbol{f}}^{C})$ as the energy conserved flux. Again, this method is a central and requires artificial dissipation to be stabilised numerically.

Recall the simpler scalar artificial signals

$$\phi_p^{\text{art}} = -\alpha[u]_{ji} - \beta[u]_{kj} - \gamma[u]_{ik}. \tag{19}$$

However, for systems of equations, the Jacobians are decomposed into their eigenvalues and eigenvectors along the streamline direction to obtain multidimensionality. The upwinded signals distribution are constructed with characteristic waves based on the difference in energy variables $\boldsymbol{v}$ according to [Barth, 1999].

Hence, the artificial signals become

$$\phi_p^{\text{art}} = -\hat{\boldsymbol{R}}\hat{\boldsymbol{D}}_\alpha\hat{\boldsymbol{R}}^T\left[\boldsymbol{v}\right]_{ji} - \hat{\boldsymbol{R}}\hat{\boldsymbol{D}}_\beta\hat{\boldsymbol{R}}^T\left[\boldsymbol{v}\right]_{kj} - \hat{\boldsymbol{R}}\hat{\boldsymbol{D}}_\gamma\hat{\boldsymbol{R}}^T\left[\boldsymbol{v}\right]_{ik}, \quad (20)$$

where

$$\hat{\boldsymbol{D}}_\alpha = \boldsymbol{\alpha}|\hat{\Lambda}|\hat{\boldsymbol{S}}l_{\mathbb{E}}, \quad \hat{\boldsymbol{D}}_\beta = \boldsymbol{\beta}|\hat{\Lambda}|\hat{\boldsymbol{S}}l_{\mathbb{E}}, \quad \hat{\boldsymbol{D}}_\gamma = \boldsymbol{\gamma}|\hat{\Lambda}|\hat{\boldsymbol{S}}l_{\mathbb{E}}, \quad (21)$$

and

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}, \quad (22)$$

with similar notations for β and γ.

After adding the artificial signal term, the energy dissipation equation becomes

$$\begin{aligned}\dot{E} &= \sum_p \vec{\boldsymbol{F}}_p \cdot \vec{n}_p - \sum_p \boldsymbol{v}_p^T \phi_p \\ &= \underbrace{\sum_p \vec{\boldsymbol{F}}_p \cdot \vec{n}_p - \sum_p \boldsymbol{v}_p^T \phi_p^{\text{iso}}}_{=0} - \sum_p \boldsymbol{v}_p^T \phi_p^{\text{art}} = -\sum_p \boldsymbol{v}_p^T \phi_p^{\text{art}}.\end{aligned} \tag{23}$$

In order to fulfil energy stability, $\dot{E} \leq 0$.
After some algebra, the following conditions are obtain

$$\alpha_d \geq 0, \quad \beta_d = 0, \quad \gamma_d = -\alpha_d, \qquad d = 1, 2, 3. \tag{24}$$

The construction of the first order scheme is based on the positivity property. Assuming $\gamma = -\alpha$ and $\beta = 0$, the signal for node i is

$$\phi_i = \frac{1}{2}\left(\vec{\boldsymbol{f}}_i - \vec{\boldsymbol{f}^C}\right)\cdot\vec{n}_i + \left(\alpha \boldsymbol{R}\Lambda\boldsymbol{R}^{-1}\right)_{\text{avg}}\left(2\boldsymbol{u}_i - \boldsymbol{u}_j - \boldsymbol{u}_k\right) \tag{25}$$

The upwind parameter matrix and conservative flux are defined as

$$\boldsymbol{K}_i = \frac{1}{2}\left(\boldsymbol{R}\vec{\Lambda}\boldsymbol{R}^{-1}\right)\cdot\vec{n}_i, \qquad \vec{\boldsymbol{f}^C} = \left(\boldsymbol{R}\vec{\Lambda}\boldsymbol{R}^{-1}\right)\bar{\boldsymbol{u}}. \tag{26}$$

Then,

$$\phi_i = \left(\tfrac{1}{3} (\boldsymbol{K})_i + \left(\alpha \boldsymbol{R} \vec{\Lambda} \boldsymbol{R}^{-1} \right)_{\text{avg}} \right) (2\boldsymbol{u}_i - \boldsymbol{u}_j - \boldsymbol{u}_k) \qquad (27)$$

In order to produce a positive solution, the condition on α is

$$\left(\alpha \boldsymbol{R} \vec{\Lambda} \boldsymbol{R}^{-1} \right)_{\text{avg}} > -\frac{1}{3} \min \left(\boldsymbol{K}_p \right), \quad p = i, j, k. \qquad (28)$$

where the α determined with this method is deemed α_{Kmin}.

Baseline Approach

In order to increase the dependency of the error on the grid size, the α can be altered as

$$\alpha_{\text{baseline}} = \begin{bmatrix} \left(\frac{l_{\mathbb{E}}}{L_r}\right)^q & 0 & 0 \\ 0 & \left(\frac{l_{\mathbb{E}}}{L_r}\right)^q & 0 \\ 0 & 0 & \left(\frac{l_{\mathbb{E}}}{L_r}\right)^q \end{bmatrix}. \tag{29}$$

where $l_{\mathbb{E}}$ is,

$$l_{\mathbb{E}} = \frac{1}{3}\sum_p l_p, \quad p = i, j, k. \tag{30}$$

Alternative Approach

Alternatively, we can use the determined α_{Kmin} from first order method and increase the dependency on grid by multiplying α_{baseline} to alter the artificial signals. It should be noted that the positivity feature no longer holds. These approaches result in solution with $(q+1)$th order accuracy by examining Eq. (21).

The existing ESRD is not well-balanced. Slight tinkering was performed on the ESRD scheme to achieve this property. The gravitational pressure term is split from the convective flux and combined with the source term.

$$\vec{\boldsymbol{f}}(\boldsymbol{u}) = \begin{bmatrix} hu & hv \\ hu^2 + \boxed{\frac{1}{2}gh^2} & huv \\ huv & hv^2 + \boxed{\frac{1}{2}gh^2} \end{bmatrix} \tag{31}$$

which leads to the well-balanced version of isotropic signal with the lake at rest conditions, $u = v = 0$, $h + z_b = \text{constant}$

$$
\begin{aligned}
\phi^{\mathbb{E},\text{iso}} = & \frac{1}{2}\sum_p \begin{bmatrix} (h_p u_p) n_{p,x} + (h_p v_p) n_{p,y} \\ (h_p u_p^2) n_{p,x} + (h_p u_p v_p) n_{p,y} \\ (h_p u_p v_p) n_{p,x} + (h_p v_p^2) n_{p,y} \end{bmatrix} \\
& + \frac{g\bar{h}}{2}\sum_p \begin{bmatrix} 0 \\ (h_p + z_{b,p}) n_{p,x} \\ (h_p + z_{b,p}) n_{p,y} \end{bmatrix} \\
& - \frac{1}{2}\sum_p \begin{bmatrix} (h^* u^*) n_{p,x} + (h^* v^*) n_{p,y} \\ (h^* (u^*)^2) n_{p,x} + (h^* u^* v^*) n_{p,y} \\ (h^* u^* v^*) n_{p,x} + (h^* (v^*)^2) n_{p,y} \end{bmatrix} \\
& - \frac{g\bar{h}}{2}\sum_p \begin{bmatrix} 0 \\ (h^* + z_b^*) n_{p,x} \\ (h^* + z_b^*) n_{p,y} \end{bmatrix} \qquad (32)
\end{aligned}
$$

For the artificial signal,

$$\phi_i = \phi_i^{\text{iso}} \underbrace{- \hat{\boldsymbol{R}}\hat{\boldsymbol{D}}_\alpha \hat{\boldsymbol{R}}^T [\boldsymbol{v}]_{ji} - \hat{\boldsymbol{R}}\hat{\boldsymbol{D}}_\beta \hat{\boldsymbol{R}}^T [\boldsymbol{v}]_{kj} - \hat{\boldsymbol{R}}\hat{\boldsymbol{D}}_\gamma \hat{\boldsymbol{R}}^T [\boldsymbol{v}]_{ik}}_{\phi_i^{\text{art}}}, \tag{33}$$

Applying the lake at rest conditions, $u = v = 0$, $h + z_b =$ constant to the discrete variables in $\boldsymbol{v}$ for an edge leads to

$$[\boldsymbol{v}]_{ji} = \begin{bmatrix} -\frac{1}{2}(u_j^2 + v_j^2 - u_i^2 - v_i^2) + g(h_j + z_{b,j} - h_i - z_{b,i}) \\ u_j - u_i \\ v_j - v_i \end{bmatrix} = 0. \tag{34}$$

Isotropic grids used

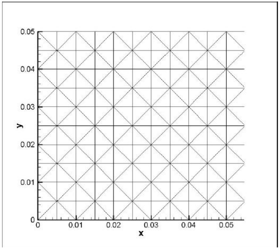


Figure: Regular

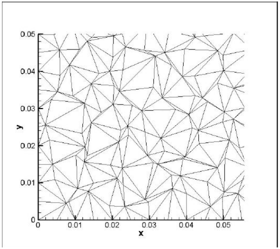
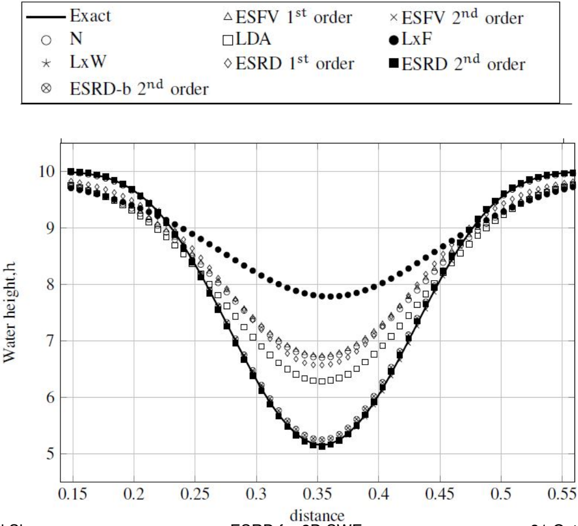
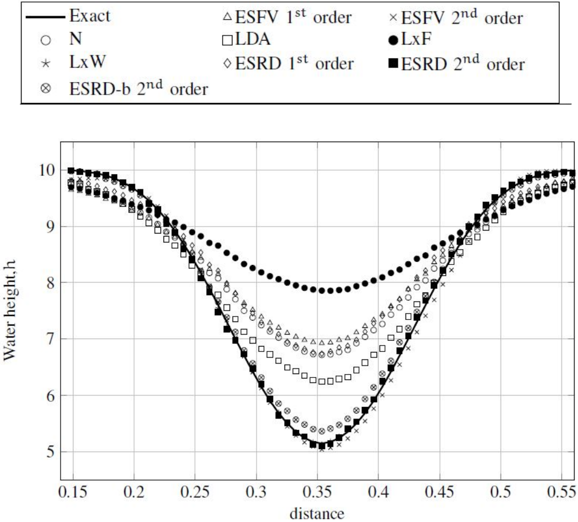


Figure: 80% randomised

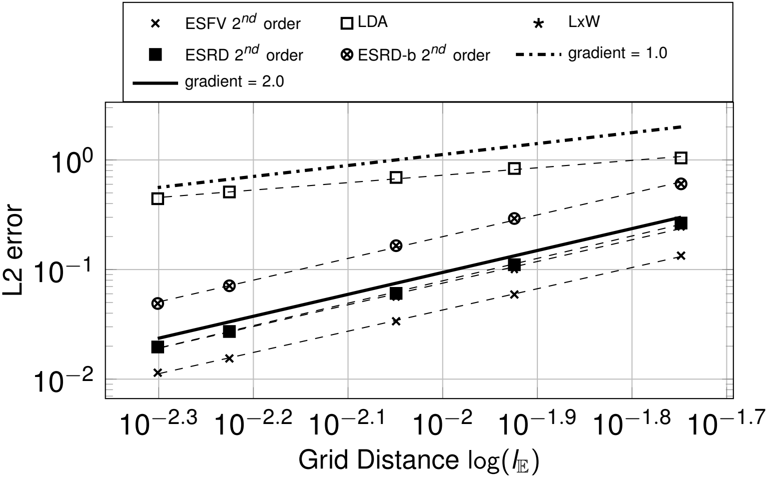
Water height profiles for regular grid



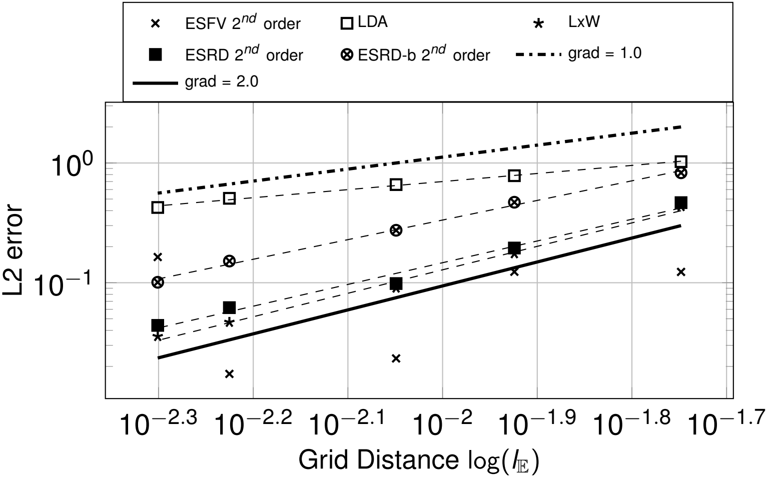
Water height profiles for 80% randomised grid



Order of Accuracy for regular grid



Order of Accuracy for 80% randomised grid



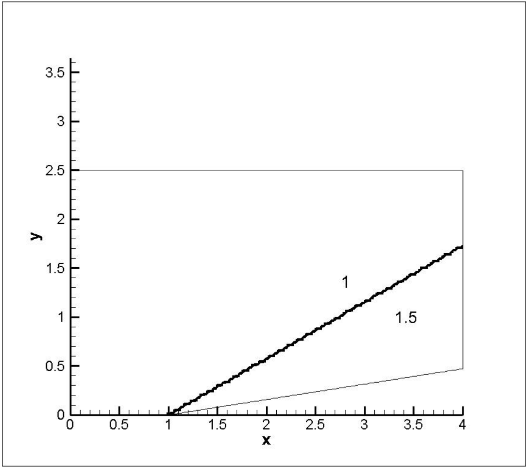


Figure: Exact solution for oblique hydraulic jump.

Results

Oblique Hydraulic Jump

Water height profiles across $y = 0.85$

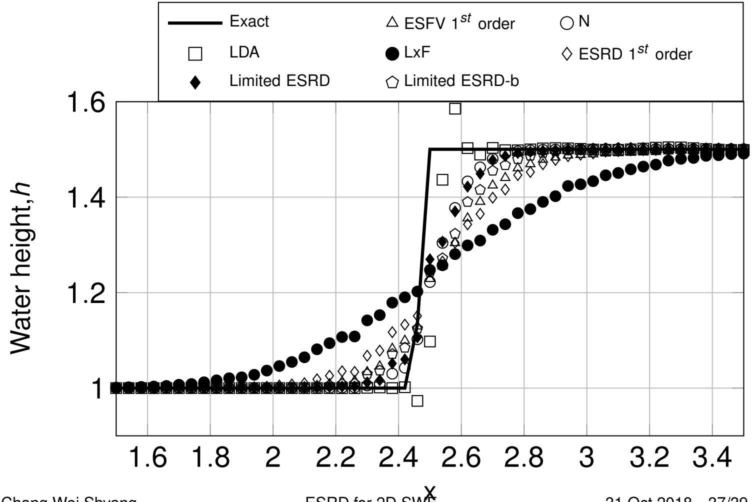


Table: Norm of the errors on water heights at $t = 0.15$.

Schemes	Average error	$L2$ error	$L\infty$ error
N	1.09×10^{-1}	1.75×10^{-1}	8.49×10^{-1}
ESFV 1^{st}	2.32×10^{-2}	4.19×10^{-2}	2.85×10^{-1}
ESRD 1^{st}	8.40×10^{-17}	1.54×10^{-16}	8.88×10^{-16}
ESRD 2^{nd}	1.96×10^{-16}	3.14×10^{-16}	1.55×10^{-15}

Thank You

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