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Entropy Stable Residual Distribution Method

Scalar Equation

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A physical conservation law could be written in,

$$\partial_t u + \partial_x f + \partial_y g = 0 \tag{1}$$

The second law of thermodynamics will be used as an additional conservation law in the differential form,

$$\partial_t U + \partial_x F + \partial_y G \leq 0 \tag{2}$$

where,

$$U = -\rho h(s), \quad F = -\rho u h(s), \quad G = -\rho v h(s) \tag{3}$$

The Fundamental of Entropy Control

Main Consept

There are three keywords here:

- Entropy Conservation: The entropy will not be increased or decreased.
- Entropy Stability: The entropy will always be generated with physical (correct) sign.
- Entropy Consistency: The entropy will always be generated with physical (correct) sign and amount.

The Fundamental of Entropy Control

Main Consept

Consider a new signal distribution for RD methods,

$$\phi_i = ((f_i, g_i) - (f^*, g^*)) \cdot \vec{n}_i$$

The entropy conservation enforce that,

$$\sum_i v_i^T \phi_i = \sum_i (F_i, G_i) \cdot \vec{n}_i$$

Or,

$$\sum_i (v_i^T (f_i, g_i) \cdot \vec{n}_i) - (f^*, g^*) \sum_i (v_i^T \vec{n}_i) = \sum_i (F_i, G_i) \cdot \vec{n}_i$$

We propose an artificial terms which alters the signals distribution in the following form.

$$\phi_i = \frac{1}{2}(f_i - f^*, g_i - g^*) \cdot \vec{n}_i$$
$$- \alpha(v_j - v_i) - \beta(v_k - v_j) - \gamma(v_i - v_k) \quad (4)$$

where (j, k) are the neighboring points to node i and that are to be determined.

The entropy generation will be,

$$\dot{U} = \sum_i (F_i, G_i) \cdot \vec{n}_i - \sum_i v_i^T (f_i, g_i) \cdot \vec{n}_i + (f^*, g^*) \sum_i (v_i^T \vec{n}_i)$$

$$+ \sum_i v_i^T (\alpha(v_j - v_i) + \beta(v_k - v_j) + \gamma(v_i - v_k))$$

$$= - \sum_i v_i^T (\alpha(v_i - v_j) + \beta(v_j - v_k) + \gamma(v_k - v_i))$$

$$= - \frac{\alpha - \gamma}{2}((v_1 - v_2)^2 + (v_2 - v_3)^2 + (v_3 - v_1)^2)$$

$$- \beta(v_1 v_2 - v_2 v_1 + v_2 v_3 - v_3 v_2 + v_3 v_1 - v_1 v_3) \leq 0$$

$$\alpha > \gamma, \quad \forall \beta \quad \rightarrow \quad \alpha > 0, \quad \gamma = -\alpha, \quad \forall \beta$$

Recovery of N-scheme

Entropy Stabel

In order to have an overview about the α and β we match the formulation to classical N approach. ($v_{ij} = v_i - v_j$)

$$\alpha = \begin{cases} \dfrac{(v_{ij} + v_{ik})\phi^T}{6\left(v_{ij}^2 + v_{jk}^2 + v_{ki}^2\right)} & \text{Two-Target} \\ \dfrac{k_j v_{ji}(v_{ij} + v_{kj}) + k_k v_{ki}(v_{ik} + v_{jk})}{6\left(v_{ij}^2 + v_{jk}^2 + v_{ki}^2\right)} & \text{One-Target} \end{cases}$$

$$\beta = \begin{cases} \dfrac{(v_{ji} + v_{ki})(k_i v_{jk} + k_j v_{ki} + k_k v_{ij})}{6\left(v_{ij}^2 + v_{jk}^2 + v_{ki}^2\right)} & \text{Two-Target} \\ \left(\dfrac{k_k - k_j}{12}\right)\left(1 - \dfrac{6v_{jk}^2}{v_{ij}^2 + v_{jk}^2 + v_{ki}^2}\right) & \text{One-Target} \end{cases}$$

Recovery of N-scheme
Entropy Consistent

For a linear case the N-scheme and LDA has this relation,

$$\phi_i^{\mathrm{N}} = \phi_i^{\mathrm{LDA}} + K_i^{+}(u_i - \hat{u}^{+}), \quad \hat{u}^{+} = \frac{\sum_p k_p^{+} u_p}{\sum_p k_p^{+}}$$

$$\phi_i^{\mathrm{N(m)}} = \phi_i^{\mathrm{LDA}} + K_i^{+}(u_i - \hat{u}^{+}) + K_i^{+,m}(u_i - \hat{u}^{+,m})$$
$$K_i^{+,m} = \kappa \vec{\lambda} \cdot \vec{n}_i$$

$$\phi_i^{\mathrm{N(m)}} = \phi_i^{\mathrm{N}} + K_i^{+,m}(u_i - \hat{u}^{+,m}),$$

The amount of entropy generation based on calculation of α for a two target cell will be,

$$\alpha^m = \kappa \left(\frac{k_j k_k v_{jk}^2}{2\left(k_j + k_k\right)\left(v_{ij}^2 + v_{jk}^2 + v_{ki}^2\right)} \right) > 0$$

$$\alpha^{\mathrm{N}} = \frac{(v_{ij} + v_{ik})\phi^T}{6\left(v_{ij}^2 + v_{jk}^2 + v_{ki}^2\right)}$$

$$\alpha^{\mathrm{N(m)}} = \alpha^{\mathrm{N}} + \alpha^m$$

Recovery of PSI

Minmod Limiter

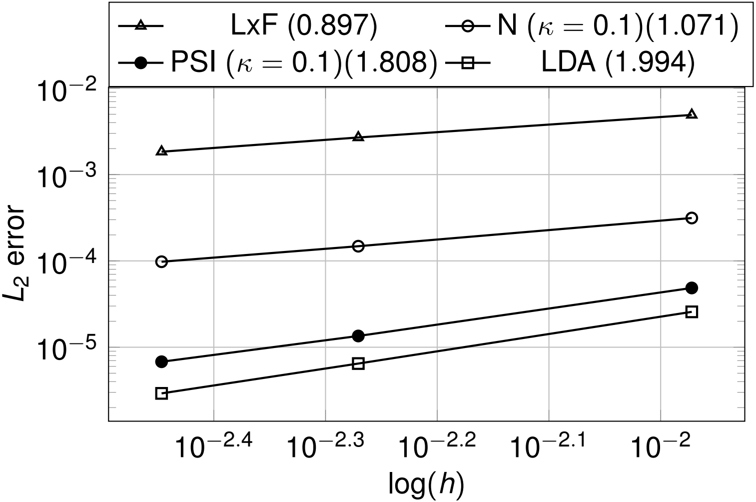
For entropy approach of scalar equation minmod limiter could be applied. Therefore,

$$\Omega_i^{\mathrm{LMT}} = \frac{\max(0, \Omega_i)}{\sum_p \max(0, \Omega_p)} \qquad \Omega_i = \frac{\phi_i}{\phi^{\mathrm{total}}} \tag{5}$$

This limiter could implement on the N and Lax-Friedrichs approaches.

Numerical Results

Order of Accuracy



Thank You

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