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Entropy Stable Residual Distribution Method

Scalar Equation

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The Fundamental of Entropy Control
Main Consept

There are three keywords here:

- Entropy Conservation: The entropy will not be increased or decreased.
- Entropy Stability: The entropy will always be generated with physical (correct) sign.
- Entropy Consistency: The entropy will always be generated with physical (correct) sign and amount.

Consider a new signal distribution for RD methods,

$$\phi_i = \frac{1}{2}\left((f_i, g_i) - (f^*, g^*)\right) \cdot \vec{n}_i$$

The entropy conservation enforce that,

$$\sum_i v_i^T \phi_i = \frac{1}{2} \sum_i (F_i, G_i) \cdot \vec{n}_i$$

Or,

$$\sum_i (v_i^T (f_i, g_i) \cdot \vec{n}_i) - (f^*, g^*) \sum_i (v_i^T \vec{n}_i) = \sum_i (F_i, G_i) \cdot \vec{n}_i$$

We propose an artificial terms which alters the signals distribution in the following form.

$$\phi_i = \frac{1}{2}\left((f_i, g_i) - (f^*, g^*)\right) \cdot \vec{n}_i \\ - \alpha(v_j - v_i) - \beta(v_k - v_j) - \gamma(v_i - v_k)$$

where (j, k) are the neighboring points to node i and that are to be determined.

$$
\begin{aligned}
\dot{U} =& \frac{1}{2}\sum_i (F_i, G_i)\cdot \vec{n}_i - \frac{1}{2}\sum_i v_i^T (f_i, g_i)\cdot \vec{n}_i \\
&+ \frac{1}{2}(f^*, g^*)\sum_i (v_i^T \vec{n}_i) \\
&+ \sum_i v_i^T(\alpha(v_j - v_i) + \beta(v_k - v_j) + \gamma(v_i - v_k)) \\
=& -\sum_i v_i^T(\alpha(v_i - v_j) + \beta(v_j - v_k) + \gamma(v_k - v_i)) \\
=& -\frac{\alpha - \gamma}{2}((v_1 - v_2)^2 + (v_2 - v_3)^2 + (v_3 - v_1)^2) \\
&- \beta(v_1 v_2 - v_2 v_1 + v_2 v_3 - v_3 v_2 + v_3 v_1 - v_1 v_3) \le 0
\end{aligned}
$$

$$\alpha > \gamma, \quad \forall\beta \quad \rightarrow \quad \alpha > 0, \quad \gamma = -\alpha, \quad \forall\beta$$

N-scheme Recovery

Determined α and β

$$\alpha = \begin{cases} \dfrac{2([v]_{ji}+[v]_{ki})\phi_T}{3\left([v]_{ij}^2+[v]_{jk}^2+[v]_{ki}^2\right)} & \text{Two-Target} \\ \dfrac{2\phi_T(4v_i+v_j+v_k)-3\left(k_i v_i^2+k_j v_j^2+k_k v_k^2\right)}{3\left([v]_{ij}^2+[v]_{jk}^2+[v]_{ki}^2\right)} & \text{One-Target} \end{cases}$$

$$\beta = \begin{cases} \dfrac{k_j-k_k}{6} + \dfrac{k_j[v]_{ik}^2-k_k[v]_{ij}^2}{[v]_{ij}^2+[v]_{jk}^2+[v]_{ki}^2} & \text{Two-Target} \\ \left(\dfrac{k_j-k_k}{6}\right)\left(\dfrac{6[v]_{jk}^2}{[v]_{ij}^2+[v]_{jk}^2+[v]_{ki}^2}-1\right) & \text{One-Target} \end{cases}$$

N-scheme will be recovered as follow,

Two-Target
$k_i < 0, \quad k_j, k_k > 0,$

$$\begin{cases} \tilde{\phi}_i^{\mathrm{N}} = 0 \\ \tilde{\phi}_j^{\mathrm{N}} = k_j^+(u_j - u_i) \\ \tilde{\phi}_k^{\mathrm{N}} = k_k^+(u_k - u_i) \end{cases}$$

One-Target
$k_i > 0, \quad k_j, k_k < 0,$

$$\begin{cases} \tilde{\phi}_i^{\mathrm{N}} = k_i u_i + k_j u_j + k_k u_k \\ \tilde{\phi}_j^{\mathrm{N}} = 0 \\ \tilde{\phi}_k^{\mathrm{N}} = 0 \end{cases}$$

In order to simplify the formulation for α and β, define,

$$\delta_{i,j,k} = v_{i,j,k} - \bar{v}$$

Therefore,

$$v_j = v_i + \delta_k + 2\delta_j, \quad v_k = v_i + 2\delta_k + \delta_j$$

Assuming $\delta_j \simeq \delta_k$,

Method	α	β
Lax-Friedrichs	$\frac{1}{6}\left(k_k + k_j + 2k_{\max}\right)$	$\frac{1}{6}\left(k_k - k_j\right)$
N (two-target)	$\frac{1}{3}\left(k_j + k_k\right)$	$\frac{1}{3}\left(k_j - k_k\right)$
LDA (two-target)	$\frac{1}{3}\left(k_j + k_k\right)$	$\frac{1}{3}\left(k_j - k_k\right)$
One-Target	$-\frac{1}{6}\left(k_j + k_k\right)$	$-\frac{1}{6}\left(k_j - k_k\right)$

Positivity Check
General α and β

Recall the linear signal distribution,

$$\begin{cases} \tilde{\phi}_i = \frac{1}{3}k_i([u]_{ij} - [u]_{ki}) - \alpha\left([u]_{ji} - [u]_{ik}\right) - \beta[u]_{kj} \\ \tilde{\phi}_j = \frac{1}{3}k_j([u]_{jk} - [u]_{ij}) - \alpha\left([u]_{kj} - [u]_{ji}\right) - \beta[u]_{ik} \\ \tilde{\phi}_k = \frac{1}{3}k_k([u]_{ki} - [u]_{jk}) - \alpha\left([u]_{ik} - [u]_{kj}\right) - \beta[u]_{ji} \end{cases}$$

Simplifying,

$$\begin{cases} \tilde{\phi}_i = \left(\frac{2}{3}k_i + 2\alpha\right)u_i + \left(-\frac{1}{3}k_i - \alpha + \beta\right)u_j + \left(-\frac{1}{3}k_i - \alpha - \beta\right)u_k \\ \tilde{\phi}_j = \left(-\frac{1}{3}k_j - \alpha - \beta\right)u_i + \left(\frac{2}{3}k_j + 2\alpha\right)u_j + \left(-\frac{1}{3}k_j - \alpha + \beta\right)u_k \\ \tilde{\phi}_k = \left(-\frac{1}{3}k_k - \alpha + \beta\right)u_i + \left(-\frac{1}{3}k_k - \alpha - \beta\right)u_j + \left(\frac{2}{3}k_k + 2\alpha\right)u_k \end{cases}$$

Thus, the positivity condition will be,

$$\alpha > -\frac{1}{3}k_{\min}, \qquad |\beta| < \alpha + \frac{1}{3}k_{\min} \tag{1}$$

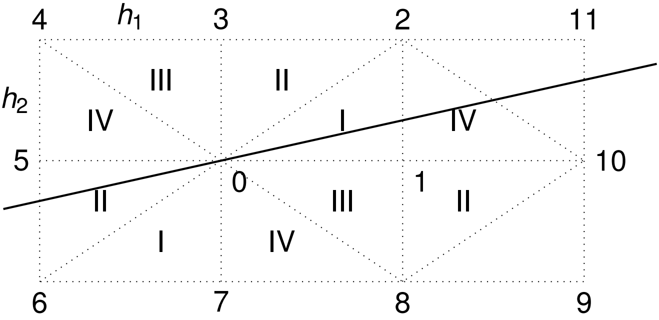
Order of Accuracy

Isotropic Grid

$$\text{TE} = \text{TE}^{\text{iso}} + \text{TE}^{\text{art}} = O\left(h^3\right) + \text{TE}^{\text{art}}$$

The geometrical dimensions are $(h_1 = h, h_2 = hs)$.
Assume,

$$\alpha = \tilde{\alpha} h, \quad \beta = \tilde{\beta} h, \quad \gamma = \tilde{\gamma} h$$



Order of Accuracy

Isotropic Grid

There are four types of triangle (I, II, III, VI). For each type α, β and γ are same. Thus, the truncation error is,

$$
\begin{aligned}
\mathrm{TE} = \; & \Big(-\left(\tilde{\alpha}_{\mathrm{I}} + \tilde{\alpha}_{\mathrm{II}} + \tilde{\alpha}_{\mathrm{III}} + \tilde{\alpha}_{\mathrm{IV}} - \tilde{\gamma}_{\mathrm{I}} - \tilde{\gamma}_{\mathrm{II}} - \tilde{\gamma}_{\mathrm{III}} - \tilde{\gamma}_{\mathrm{IV}}\right) a^2 s^2 \\
& + \left(2\tilde{\alpha}_{\mathrm{I}} - \tilde{\alpha}_{\mathrm{III}} - \tilde{\alpha}_{\mathrm{IV}} - \tilde{\beta}_{\mathrm{I}} + \tilde{\beta}_{\mathrm{II}} - \tilde{\beta}_{\mathrm{III}} + \tilde{\beta}_{\mathrm{IV}} - \tilde{\gamma}_{\mathrm{I}} - \tilde{\gamma}_{\mathrm{II}} + 2\tilde{\gamma}_{\mathrm{III}}\right) abs \\
& - \left(\tilde{\alpha}_{\mathrm{I}} + \tilde{\alpha}_{\mathrm{II}} + 2\tilde{\alpha}_{\mathrm{IV}} - \tilde{\beta}_{\mathrm{I}} + \tilde{\beta}_{\mathrm{II}} + \tilde{\beta}_{\mathrm{III}} - \tilde{\beta}_{\mathrm{IV}} - 2\tilde{\gamma}_{\mathrm{II}} - \tilde{\gamma}_{\mathrm{III}} - \tilde{\gamma}_{\mathrm{IV}}\right) b^2 \Big) \left(\frac{u_{nn}}{2r^2 s}\right) h \\
& + \Big(2b^3\left(\tilde{\alpha}_{\mathrm{IV}} + \tilde{\beta}_{\mathrm{II}} - \tilde{\beta}_{\mathrm{IV}} - \tilde{\gamma}_{\mathrm{II}}\right) + ab^2 s\left(\tilde{\alpha}_{\mathrm{III}} - \tilde{\alpha}_{\mathrm{IV}} - \tilde{\beta}_{\mathrm{I}} + \tilde{\beta}_{\mathrm{II}} - \tilde{\beta}_{\mathrm{III}} + \tilde{\beta}_{\mathrm{IV}} + \tilde{\gamma}_{\mathrm{I}} - \tilde{\gamma}_{\mathrm{II}}\right) \\
& + a^2 bs^2\left(2\tilde{\alpha}_{\mathrm{II}} + \tilde{\alpha}_{\mathrm{III}} - \tilde{\alpha}_{\mathrm{IV}} + \tilde{\beta}_{\mathrm{I}} - 3\tilde{\beta}_{\mathrm{II}} - \tilde{\beta}_{\mathrm{III}} + 3\tilde{\beta}_{\mathrm{IV}} - \tilde{\gamma}_{\mathrm{I}} + \tilde{\gamma}_{\mathrm{II}} - 2\tilde{\gamma}_{\mathrm{IV}}\right)\Big) \left(\frac{u_{nnn}}{4r^3 s}\right) h^2 \\
& + O\left(h^3\right)
\end{aligned}
$$

High order way,

$$
\alpha, \beta, \gamma = O\left(h^q\right) \Rightarrow \text{Order of Accuracy} = q + 1
$$

Minmod limiter could be applied as,

$$\Omega_i^{\text{LMT}} = \frac{\max(0, \Omega_i)}{\sum_p \max(0, \Omega_p)}, \qquad \Omega_i = \frac{\phi_i}{\phi_T}$$

And,

$$\phi_i^{\text{LMT}} = \Omega_i^{\text{LMT}} \phi_T$$

This limiter could implement on the N and Lax-Friedrichs approaches. There are three possibilities for the $\Omega_{i,j,k}$:

$$\Omega_{i,j,k} \geq 0, \quad \Omega_i < 0, \Omega_{j,k} \geq 0, \quad \Omega_i \geq 0, \Omega_{j,k} < 0.$$

Signal Limiter For α and β

All Positive

All positive, $\Omega_i, \Omega_j, \Omega_k \geq 0$ which is going back to the original signal distribution. Because the LP is satisfied in this condition automatically.

$$\Omega_{i,j,k} = \frac{\phi_{i,j,k}}{\phi_T}$$

Signal Limiter For α and β

Two Positive

Two positive, $\Omega_i < 0$ and $\Omega_j, \Omega_k \geq 0$ which leads to,

$$\begin{cases} \tilde{\phi}_i^{\text{LMT}} &= (1-p)\tilde{\phi}_i \\ \tilde{\phi}_j^{\text{LMT}} &= \tilde{\phi}_j + p\tilde{\phi}_i\Omega_j^{\text{LMT}} \\ \tilde{\phi}_k^{\text{LMT}} &= \tilde{\phi}_k + p\tilde{\phi}_i\Omega_k^{\text{LMT}} \end{cases}, \quad 0 \leq p \leq 1, \quad \Omega_j^{\text{LMT}} + \Omega_k^{\text{LMT}} = 1$$

For $p = 1$ the limiter will reduce to the classic minmod.

$$\begin{cases} \tilde{\phi}_i = \left(\frac{2}{3}k_i + 2\alpha\right) u_i + \left(-\frac{1}{3}k_i - \alpha + \beta\right) u_j + \left(-\frac{1}{3}k_i - \alpha - \beta\right) u_k \\ \tilde{\phi}_j = \left(-\frac{1}{3}k_j - \alpha - \beta\right) u_i + \left(\frac{2}{3}k_j + 2\alpha\right) u_j + \left(-\frac{1}{3}k_j - \alpha + \beta\right) u_k \\ \tilde{\phi}_k = \left(-\frac{1}{3}k_k - \alpha + \beta\right) u_i + \left(-\frac{1}{3}k_k - \alpha - \beta\right) u_j + \left(\frac{2}{3}k_k + 2\alpha\right) u_k \end{cases}$$

which leads to,

$$p < \min\left[\frac{\frac{1}{3}k_j+\alpha+\beta}{\Omega_j^{\text{LMT}}\left(\frac{2}{3}k_i+2\alpha\right)}, \frac{\frac{2}{3}k_j+2\alpha}{\Omega_j^{\text{LMT}}\left(\frac{1}{3}k_i+\alpha-\beta\right)}, \frac{\frac{1}{3}k_k+\alpha-\beta}{\Omega_k^{\text{LMT}}\left(\frac{2}{3}k_i+2\alpha\right)}, \frac{\frac{2}{3}k_k+2\alpha}{\Omega_k^{\text{LMT}}\left(\frac{1}{3}k_i+\alpha+\beta\right)}\right]$$

One positive, $\Omega_j, \Omega_k < 0$ and $\Omega_i \geq 0$ that make the whole signal distribution one target after limiting,

$$\begin{cases} \tilde{\phi}_i^{\text{LMT}} = \tilde{\phi}_i + p\left(\tilde{\phi}_j + \tilde{\phi}_k\right) \\ \tilde{\phi}_j^{\text{LMT}} = (1-p)\tilde{\phi}_j \\ \tilde{\phi}_k^{\text{LMT}} = (1-p)\tilde{\phi}_k \end{cases}, \qquad 0 \leq p \leq 1$$

Note that, $p = 1$ for one positive case. Therefore,

$$\begin{cases} p < \dfrac{\frac{2}{3}k_i + 2\alpha}{-\frac{1}{3}k_i + 2\alpha} \\ p < \dfrac{\alpha - \beta + \frac{1}{3}k_i}{\alpha - \beta + k_j + \frac{1}{3}k_i}, \quad \alpha - \beta + k_j + \dfrac{1}{3}k_i > 0 \\ p < \dfrac{\alpha + \beta + \frac{1}{3}k_i}{\alpha + \beta + k_k + \frac{1}{3}k_i}, \quad \alpha + \beta + k_k + \dfrac{1}{3}k_i > 0 \end{cases}$$

Recall,

$$\alpha, \beta, \gamma = O\left(h^{q}\right) \Rightarrow \text{Order of Accuracy} = q + 1$$

Consider,

$$\alpha = O\left(h^{q}\right) = -\gamma, \quad \beta = 0$$

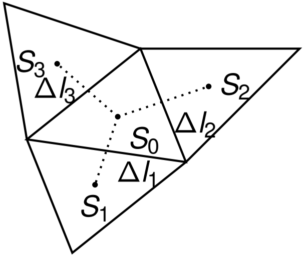
For first order we use $q = q_{\text{min}} = 0$ and for high order $q = q_{\text{max}} = 1, 2, 3, \cdots$. Therefore, by changing the q the blending approach will be constructed.

$$\alpha = \psi h^{q_{\text{min}}} + (1 - \psi) h^{q_{\text{max}}}$$

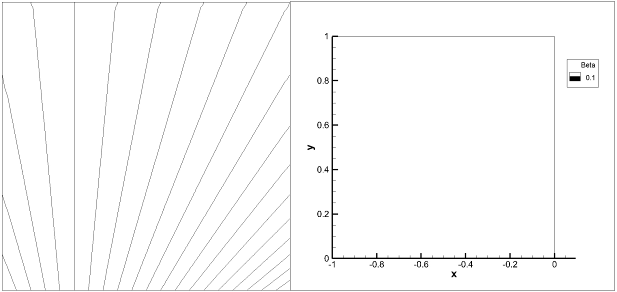
Blending Approach
Sensor Definition

$$m_i^S = \left| \frac{S_i - S_0}{\Delta l_i} \right|, \qquad m_{\max}^S = \max(m_1^S, m_2^S, m_3^S)$$

$$\psi = \begin{cases} 1 & m_{\max}^S \geq 1 \\ m_{\max}^S & m_{\max}^S < 1 \end{cases}$$

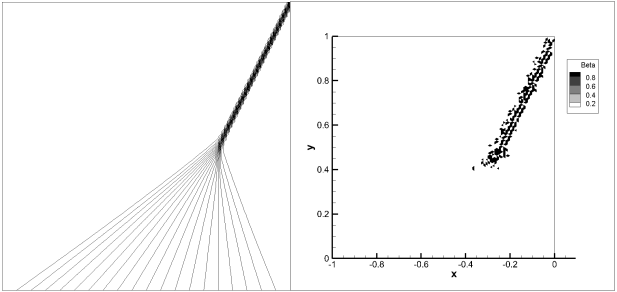


Blending Approach
Expansion



(a) Solution (b) Sensor

Blending Approach
Tree



(c) Solution (d) Sensor

Thank You

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