RD Method for Unsteady Euler's System

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From Scalar Advection Law to Euler System

- Jacobian matrix
- Inflow Matrix (positive and negative)
- Averaged Jacobian matrix

RD Method for Unsteady Linear Advection Problem

Consider the advection equation for linear PDE:

$$u_t + \vec{a} \cdot \nabla u = 0$$

This equation can be discretised for the numerical calculation using RD method as following:

$$\sum_{T \in \cup \Delta_i} \left(\underbrace{\sum_{j \in T} m_{ij}^T \frac{u_i^{n+1} - u_i^n}{\Delta t}}_{\text{unsteady residual}} + \underbrace{\beta_i \phi^T}_{\text{spatial / steady residual}} \right) = 0$$

spatial / steady residual

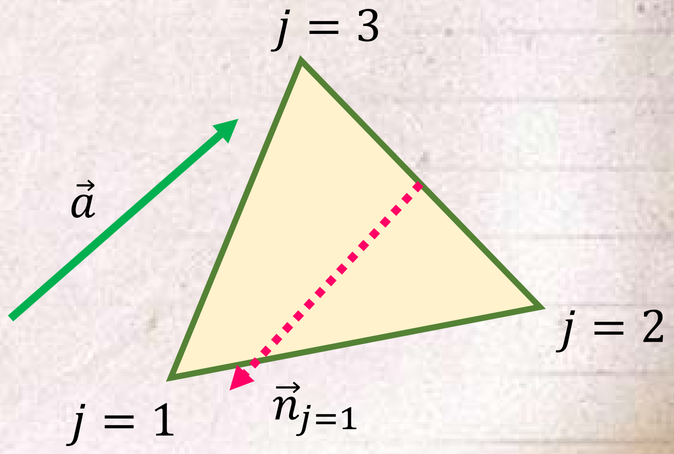
unsteady residual

The spatial residual is computed as

$$\phi^T = \sum_{j \in T} k_j u_j$$

where the inflow parameter is

$$k_j = \frac{1}{2} \vec{a} \cdot \vec{n}_j$$



General Equations of Euler's System

$$\vec{U}_t + \vec{F}_x(\vec{U}) + \vec{G}_y(\vec{U}) = 0$$

where $\vec{U}$ is the primitive variables and the fluxes $F(\vec{U})$ and $\mathrm{G}(\vec{U})$ are

$$\vec{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} \qquad \vec{F}(\vec{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix} \qquad \vec{G}(\vec{U}) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}$$

where p is the pressure

$$p = (\gamma - 1)\left[E - \frac{1}{2}\rho(u^2 + v^2)\right]$$

Euler's System is in a more general form called the *conservation law*.

Eq. (1) : conservation of mass
Eq. (2) : conservation of momentum in x-direction
Eq. (3) : conservation of momentum in y-direction
Eq. (4) : conservation of energy

In fact, advection problem is just a specific case for conservation law.

Let us link the linear advection problem to Euler's system:

Euler's system

$$\vec{U}_t + \vec{F}_x(\vec{U}) + \vec{G}_y(\vec{U}) = 0$$

$$\vec{U}_t + \frac{\partial \vec{F}(\vec{U})}{\partial \vec{U}} \frac{\partial \vec{U}}{\partial x} + \frac{\partial \vec{G}(\vec{U})}{\partial \vec{U}} \frac{\partial \vec{U}}{\partial y} = 0$$

Linear Advection

$$u_t + \vec{a} \cdot \nabla u = 0$$

$$u_t + a_x \frac{\partial u}{\partial x} + a_y \frac{\partial u}{\partial y} = 0$$

The analogy of the "fluxes' speed" in Euler's system is called the Jacobian.

The Jacobian matrix can be found as following:

$$\vec{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}$$

$$\vec{F}(\vec{U}) = \begin{pmatrix} f_1(\vec{U}) \\ f_2(\vec{U}) \\ f_3(\vec{U}) \\ f_4(\vec{U}) \end{pmatrix} \qquad \vec{G}(\vec{U}) = \begin{pmatrix} g_1(\vec{U}) \\ g_2(\vec{U}) \\ g_3(\vec{U}) \\ g_4(\vec{U}) \end{pmatrix}$$

$$\frac{\partial \vec{F}(\vec{U})}{\partial \vec{U}} = \begin{pmatrix} \frac{\partial f_1(\vec{U})}{\partial U_1} & \frac{\partial f_1(\vec{U})}{\partial U_2} & \frac{\partial f_1(\vec{U})}{\partial U_3} & \frac{\partial f_1(\vec{U})}{\partial U_4} \\ \frac{\partial f_2(\vec{U})}{\partial U_1} & \frac{\partial f_2(\vec{U})}{\partial U_2} & \frac{\partial f_2(\vec{U})}{\partial U_3} & \frac{\partial f_2(\vec{U})}{\partial U_4} \\ \frac{\partial f_3(\vec{U})}{\partial U_1} & \frac{\partial f_3(\vec{U})}{\partial U_2} & \frac{\partial f_3(\vec{U})}{\partial U_3} & \frac{\partial f_3(\vec{U})}{\partial U_4} \\ \frac{\partial f_4(\vec{U})}{\partial U_1} & \frac{\partial f_4(\vec{U})}{\partial U_2} & \frac{\partial f_4(\vec{U})}{\partial U_3} & \frac{\partial f_4(\vec{U})}{\partial U_4} \end{pmatrix}$$

Similar procedure applies for $\frac{\partial \vec{G}(\vec{U})}{\partial \vec{U}}$

Inflow Matrix K_j for Euler's System

Euler's system

inflow matrix: $K_j = \frac{1}{2} A \left|\vec{n}_j\right|$

Jacobian: $A = \frac{\partial \vec{F}(\vec{U})}{\partial \vec{U}} n_x + \frac{\partial \vec{G}(\vec{U})}{\partial \vec{U}} n_y$

Linear Advection

inflow parameter: $k_j = \frac{1}{2} \vec{a} \cdot \vec{n}_j$

Speed: $\vec{a} = a_x \hat{\imath} + a_y \hat{\jmath}$

The Jacobian matrix is given as

$$A = \begin{pmatrix} 0 & n_x & n_y & 0 \\ \frac{1}{2}(\gamma-1)q^2 n_x - u\vec{u}\cdot\vec{n} & -(\gamma-2)u n_x + \vec{u}\cdot\vec{n} & -(\gamma-1)v n_x + u n_y & (\gamma-1)n_x \\ \frac{1}{2}(\gamma-1)q^2 n_y - v\vec{u}\cdot\vec{n} & -(\gamma-1)u n_y + v n_x & -(\gamma-2)v n_y + \vec{u}\cdot\vec{n} & (\gamma-1)n_y \\ \vec{u}\cdot\vec{n}\left[\frac{1}{2}(\gamma-1)q^2 - H\right] & H n_x - (\gamma-1)u\vec{u}\cdot\vec{n} & H n_y - (\gamma-1)v\vec{u}\cdot\vec{n} & \gamma\vec{u}\cdot\vec{n} \end{pmatrix}$$

where H is the enthalpy, $H = \frac{p+E}{\rho}$

$$q^2 = u^2 + v^2$$

Diagonalisability of the Jacobian Matrix

Euler's system

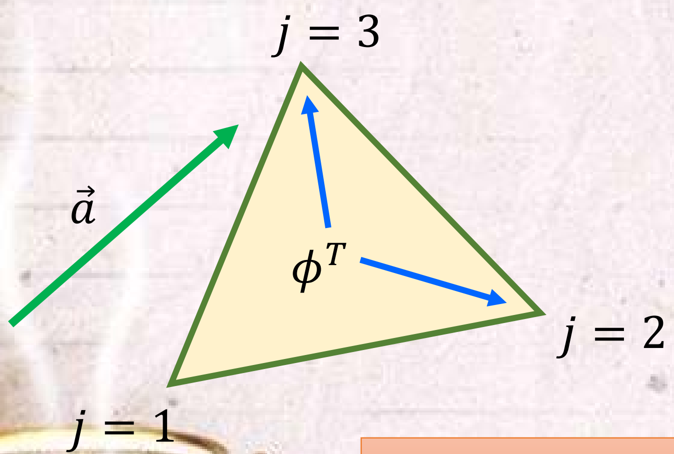
inflow matrix: $K_j = \frac{1}{2} A \left|\vec{n}_j\right|$

Linear Advection

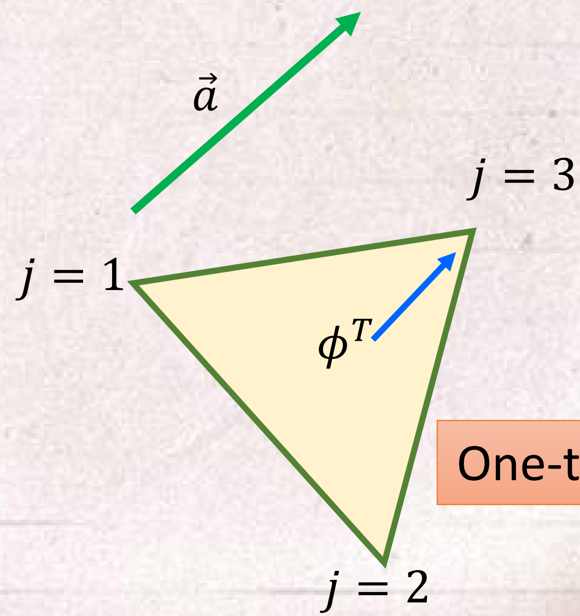
inflow parameter: $k_j = \frac{1}{2} \vec{a} \cdot \vec{n}_j$

For the LDA scheme, the distribution coefficient is given as:

$$\beta_i = \frac{k_i^+}{\sum_{j \in T} k_j^+} \qquad k_i^+ = \max(0, k_j)$$



Two-target case



One-target case

The question now is how to find the corresponding positive and negative inflow matrix for Euler's system?

$$K^{+} \quad ; \quad K^{-}$$

Euler's system is a hyperbolic PDE because its Jacobian matrix is diagonalisable

A matrix is said to be diagonalisable *iff*

$$B = C^{-1}DC$$

where D is a diagonal matrix

$$\mathrm{A} = \mathrm{R}\Lambda R^{-1}$$

A is the Jacobian,
R is the right eigenvectors,
Λ is an diagonal matrix which its diagonal entries are the eigenvalues

The Jacobian matrix can be expressed as an eigenvalue problem:

$$AR^{(i)} = \lambda_i R^{(i)}$$

There are 4 eigenvalues and therefore 4 eigenvectors, arranged in increasing magnitude of the eigenvalues give

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \qquad R = \left(\vec{R}^{(1)}, \vec{R}^{(2)}, \vec{R}^{(3)}, \vec{R}^{(4)}\right)$$

where

$$\lambda_1 = \vec{u}\cdot\vec{n} - \mathrm{c} \qquad \lambda_2 = \lambda_3 = \vec{u}\cdot\vec{n} \qquad \lambda_4 = \vec{u}\cdot\vec{n} + \mathrm{c}$$

$$R = \begin{pmatrix} 1 & 0 & 1 & 1 \\ u - cn_x & -n_y & u & u + cn_x \\ v - cn_y & n_x & v & v + cn_y \\ H - c\vec{u}\cdot\vec{n} & -un_y + vn_x & \dfrac{u^2+v^2}{2} & H + c\vec{u}\cdot\vec{n} \end{pmatrix}$$

where c is the sound speed and is given by : $c^2 = (\gamma - 1)\left[H - \frac{1}{2}q^2\right]$

R^{-1} is the inverse of the right eigenvectors or just simply known as the left eigenvectors.

Euler's system	Linear Advection
inflow matrix: $K_j = \frac{1}{2} A \left|\vec{n}_j\right|$	inflow parameter: $k_j = \frac{1}{2} \vec{a} \cdot \vec{n}_j$
$A^+ = R\Lambda^+ R^{-1}$	$k_j^+ = \begin{cases} k_j & if \quad \bar{\vec{a}} \cdot \vec{n}_j > 0 \\ 0 & if \quad \bar{\vec{a}} \cdot \vec{n}_j \leq 0 \end{cases}$
$A^- = R\Lambda^- R^{-1}$	$k_j^- = \begin{cases} k_j & if \quad \bar{\vec{a}} \cdot \vec{n}_j < 0 \\ 0 & if \quad \bar{\vec{a}} \cdot \vec{n}_j \geq 0 \end{cases}$

The distribution matrix then becomes:

$$B_i = K_i^+ \left(\sum_{j \in T} K_j^+ \right)^{-1}$$

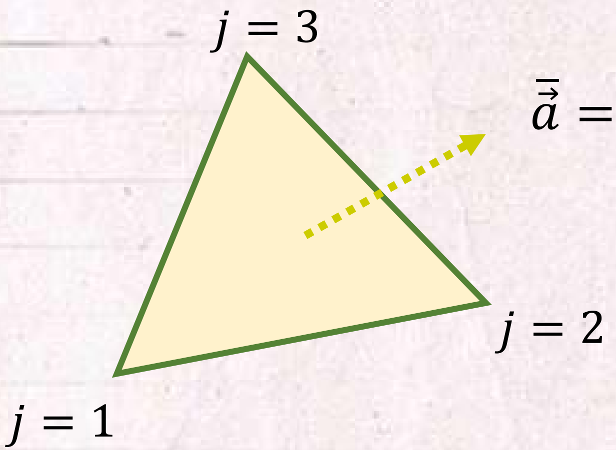
and the total spatial residual of an element is:

$$\Phi^T = \iint_T [\vec{F}_x(\vec{U}) + \vec{G}_x(\vec{U})] dxdy = \oint_T [\vec{F}(\vec{U}) dy + \vec{G}(\vec{U}) dx]$$

$$= \frac{1}{2} \sum_{j \in T} [\vec{F}(\vec{U}_j) + \vec{G}(\vec{U}_j)] \cdot \vec{n}_j$$

The Averaged Jacobian of an Element T

In scalar advection problem (both linear advection and Burger's equation), the speed vector has to be assumed constant in each element so that the RD scheme is conservative.



$$\bar{\vec{a}} = \frac{\vec{a}_1 + \vec{a}_2 + \vec{a}_3}{3}$$

This in turn implies that all the primitive variables of Jacobian A must be assumed to be constant within each element T.

Eigenvalues:

$$\lambda_1 = \vec{\bar{u}} \cdot \vec{n} - \bar{c} \qquad \lambda_2 = \lambda_3 = \vec{\bar{u}} \cdot \vec{n} \qquad \lambda_4 = \vec{\bar{u}} \cdot \vec{n} + \bar{c}$$

Right Eigenvectors:

$$R = \begin{pmatrix} 1 & 0 & 1 & 1 \\ \bar{u} - \bar{c}n_x & -n_y & \bar{u} & \bar{u} + \bar{c}n_x \\ \bar{v} - \bar{c}n_y & n_x & \bar{v} & \bar{v} + \bar{c}n_y \\ \bar{H} - \bar{c}\vec{\bar{u}} \cdot \vec{n} & -\bar{u}n_y + \bar{v}n_x & \dfrac{\bar{u}^2 + \bar{v}^2}{2} & \bar{H} + \bar{c}\vec{\bar{u}} \cdot \vec{n} \end{pmatrix}$$

First of all, find the averaged value for all the terms contained in the primitive variables:

$$\bar{\rho} = \frac{\rho_1 + \rho_2 + \rho_3}{3} \qquad \bar{u} = \frac{u_1 + u_2 + u_3}{3} \qquad \bar{v} = \frac{v_1 + v_2 + v_3}{3} \qquad \bar{E} = \frac{E_1 + E_2 + E_3}{3}$$

and the use these averaged values to compute for the following variables:

$$\bar{p} = (\gamma - 1)\left[\bar{E} - \frac{1}{2}\bar{\rho}(\bar{u}^2 + \bar{v}^2)\right] \qquad \bar{H} = \frac{\bar{p} + \bar{E}}{\bar{\rho}} \qquad \bar{q}^2 = \bar{u}^2 + \bar{v}^2$$

$$\bar{c}^2 = (\gamma - 1)\left[\bar{H} - \frac{1}{2}\bar{q}^2\right]$$

Generalising the Scalar Advection Equation to the Euler's System – the Mass Matrix

Scalar Advection

$$\sum_{T\in\cup\Delta_i}\left(\sum_{j\in T} m_{ij}^T \frac{u_i^{n+1}-u_i^n}{\Delta t} + \beta_i \phi^T\right) = 0$$

$$m_{ij}^T = \frac{S_T}{3}\beta_i\left(\delta_{ij} + 1 - \beta_j\right)$$

Euler's system

$$B_i = K_i^+\left(\sum_{j\in T} K_j^+\right)^{-1}$$

$$\Phi^T = \frac{1}{2}\sum_{j\in T}\left[\vec{F}(\vec{U}_j) + \vec{G}(\vec{U}_j)\right]\cdot\vec{n}_j$$

$$\sum_{T\in\cup\Delta_i}\left(\sum_{j\in T} M_{ij}^T \frac{\vec{U}_i^{n+1}-\vec{U}_i^n}{\Delta t} + B_i \Phi^T\right) = 0$$

identity matrix

$$M_{ij}^T = \frac{S_T}{3}B_i\left(\delta_{ij} I + I - B_j\right)$$

Remarks and Corrections for last presentation (9/3/2016):

1 The semi-discrete model for scalar advection problem reads,

$$\left[\frac{u_i^{n+1}-u_i^n}{\Delta t}+\mathcal{O}(\Delta t^2)\right]^{n+\frac{1}{2}}+[\vec{a}\cdot\nabla u]^{n+\frac{1}{2}}=0$$

The expansion of the Taylor's series at time level $n+\frac{1}{2}$ is to preserve the second order accuracy in the discretisation of time derivative.

<u>In previous presentation</u>, where the advection speed for Burgers' equation $\vec{a}(u)$ depend on u and it varies with time:

$$\sum_{T\in\cup\Delta_i}\left(\sum_{j\in T} m_{ij}^{T,n+\frac{1}{2}}\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{1}{2}\beta_i^{n+\frac{1}{2}}\left(\phi^T(u^{n+1})+\phi^T(u^n)\right)\right)=0$$

In actual case, implicit scheme with mass matrix should be implemented as follow

$$\sum_{T\in\cup\Delta_i}\left(\sum_{j\in T}\left(m_{ij}^{T,n+1}\frac{u_i^{n+1}}{\Delta t}-m_{ij}^{T,n}\frac{u_i^{n}}{\Delta t}\right)+\frac{1}{2}\left(\beta_i^{n+1}\phi^T(u^{n+1})+\beta_i^{n}\phi^T(u^{n})\right)\right)=0$$

2 The mass matrix is constructed based on the assumption of linear preserving scheme.

Examples of RD framework for unsteady case using mass matrix are:

Rossiello:
UCV, LDA, monotone FCT procedure

Ricchiuto, Abgrall :
LDA, blended LDA-N, SU, central-blended (with Lax-Friedrich's dissipation)

All these scheme are linear preserving.

Linear preservation means that the distribution coefficient must be bounded.

$$0 \leq \beta \leq 1$$

The linear preserving property of the distribution coefficient will ensure the existence of mass matrix m_{ij}^{T}

Streamline upwind mass matrix

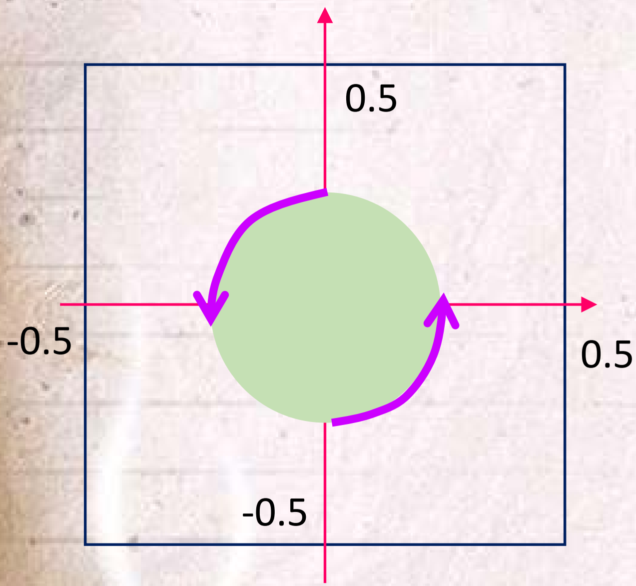
$$m_{ij}^{T} = \frac{S_T}{3}\begin{bmatrix} \beta_1^T(2-\beta_1^T) & \beta_1^T(1-\beta_2^T) & \beta_1^T(1-\beta_3^T) \\ \beta_2^T(1-\beta_1^T) & \beta_2^T(2-\beta_2^T) & \beta_2^T(1-\beta_3^T) \\ \beta_3^T(1-\beta_1^T) & \beta_3^T(1-\beta_2^T) & \beta_3^T(2-\beta_3^T) \end{bmatrix}$$

Example of Study Case:
Advection of Vortex

- Description of the test case
- Numerical results

$$\vec{U}_t + \vec{F}_x(\vec{U}) + \vec{G}_y(\vec{U}) = 0$$

$$\vec{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} \qquad \vec{F}(\vec{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix} \qquad \vec{G}(\vec{U}) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}$$



1 Density is assumed to be constant throughout the domain

$$\rho = 1.4$$

The centre of vortex is initially set to be

$$(x_c, y_c) = (0,0)$$

$$r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$$

2 The flow velocity is given as

$$\vec{v} = \vec{v}_m + \vec{v}_p$$

Mean stream velocity: $\vec{v}_m = (6,0)$

Circumferential perturbation:

$$\vec{v}_p = \begin{cases} 15(\cos(4\pi r) + 1)\big(-(y - y_c), (x - x_c)\big), & r < 0.25 \\ 0 \qquad , & otherwise \end{cases}$$

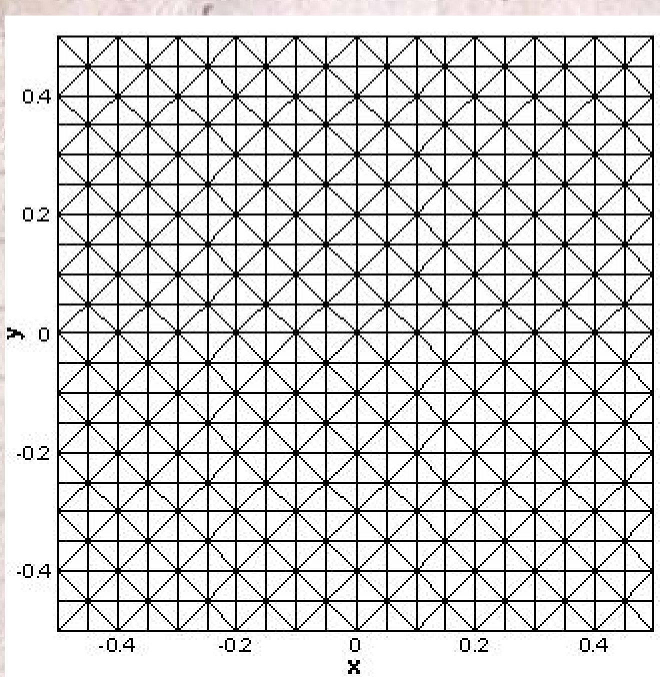
3 The pressure is given as

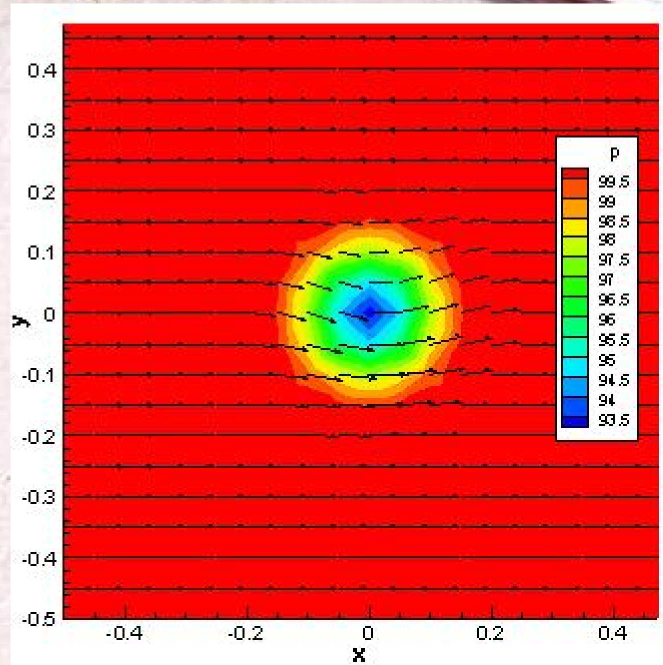
$$p = p_m + p_p$$

$$p_m = 100 \qquad p_p = \begin{cases} \Delta p \qquad , & r < 0.25 \\ 0 \qquad , & otherwise \end{cases}$$

$$\Delta p = \frac{15^2 \rho}{(4\pi)^2}\left(2\cos(4\pi r) + 8\pi r \sin(4\pi r) + \frac{\cos(8\pi r)}{8} + \pi r \sin(8\pi r) + 12\pi^2 r^2\right) + C$$

$$C = -11.02544849$$





Global time step:

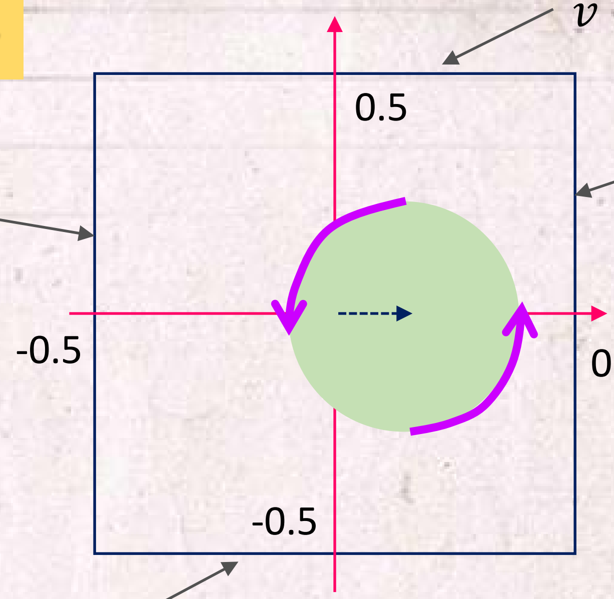
$$\Delta t = \min_i \Delta t_i$$

$$\Delta t_i = CFL \frac{S_i}{\sum_{T \in \cup \Delta_i} \alpha^T}$$

$$\alpha^T = \frac{1}{2} \max_{j \in T} \left(\| \vec{v}_j + c_j \| \right) |\vec{n}_j|$$

$$c_j = \sqrt{\frac{\gamma p_j}{\rho_j}}$$

Advection of vortex:



The centre of vortex has to be shifted as time elapsed:

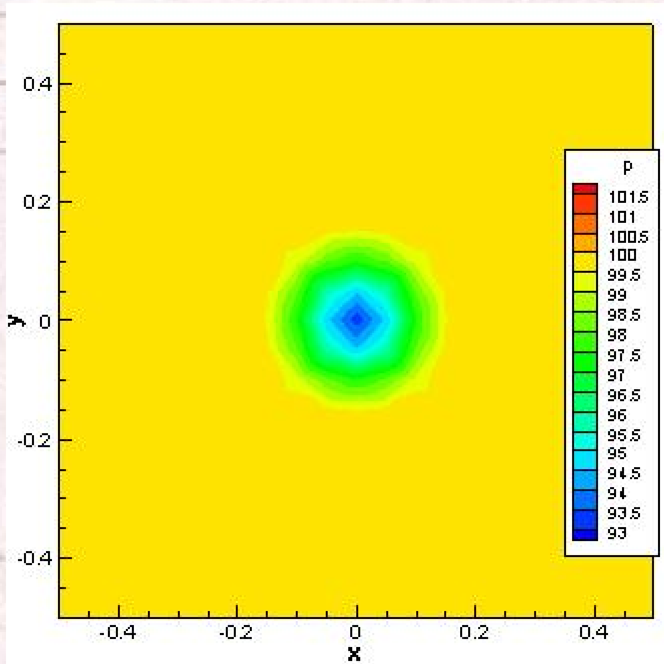
$$(x_c, y_c) = \Delta t\, \vec{v}_m$$

$$\vec{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}$$

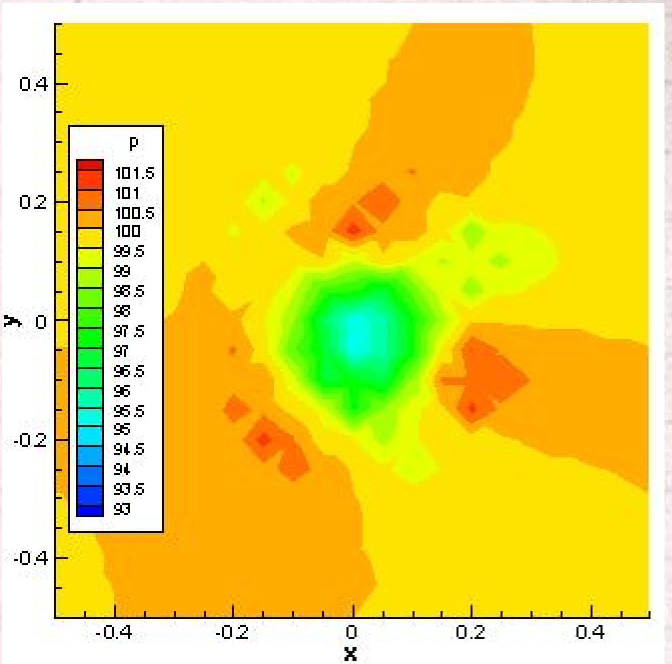
ρ, ρu, ρv → *fixed*

E → *Unknown, and the pressure is to be calculated from*

$$p = (\gamma - 1)\left[E - \frac{1}{2}\rho(u^2 + v^2)\right]$$



Exact pressure at $t = \frac{1}{6}$



Numerical results at $t = \frac{1}{6}$