Third-Order-Accurate for Scalar Conservation Laws

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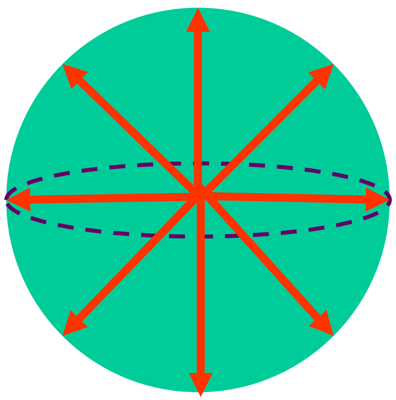
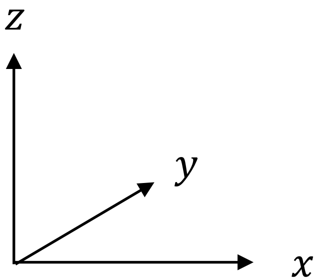
Divergence Theorem

<u>In 3-D case: **Divergence Theorem**</u>

$$\iiint_V \nabla \cdot \vec{F}\, dx\, dy\, dz = \iint_A \vec{F} \cdot \hat{n}\, dx\, dy$$

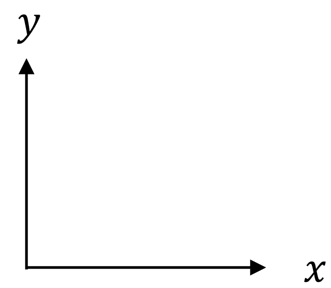
Divergence inside a volume V = fluxes that penetrate the bounded surface A





In 2-D case: Normal Form of **Green's Theorem**

$$\iint_S \nabla \cdot \vec{F} \, dx \, dy = \oint_{\partial S} \vec{F} \cdot \hat{n} \, dl$$



or commonly written in most textbooks as:

Let

$$\vec{F} = M \, \hat{\imath} + N \, \hat{\jmath}$$

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Therefore,

$$\iint_S \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy = \oint_{\partial S} \vec{F} \cdot \hat{n} \, dl$$

$$= \oint_{\partial S} M \, dy - N \, dx$$

This is commonly known as **Green's theorem**.

Examples of Exact Integration

Let us demonstrate the exact integration of flux-divergence in 2D.

Divergence Theorem

$$\iint_S \nabla \cdot \vec{F}\, dx\, dy = \oint_{\partial S} \vec{F} \cdot \hat{n}\, dl$$

Scalar Conservation Law

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{F}(\mathrm{u}) = 0$$

Problem statement

For the simplest case:

$$u(x, y) = x^2 + y^2 \qquad \vec{\lambda} = \left(a_x, a_y\right)^T$$

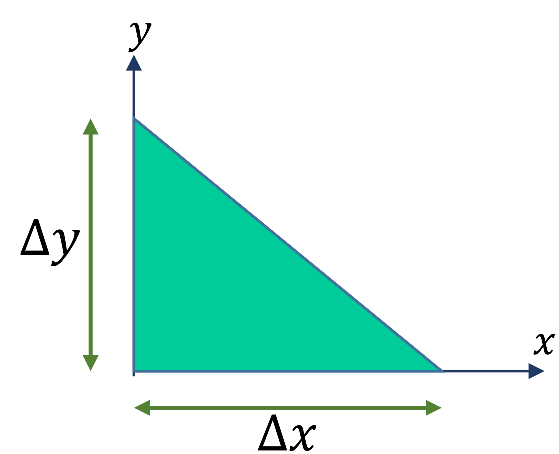
and the flux is

$$\vec{F}(\mathrm{u}) = u\vec{\lambda}$$

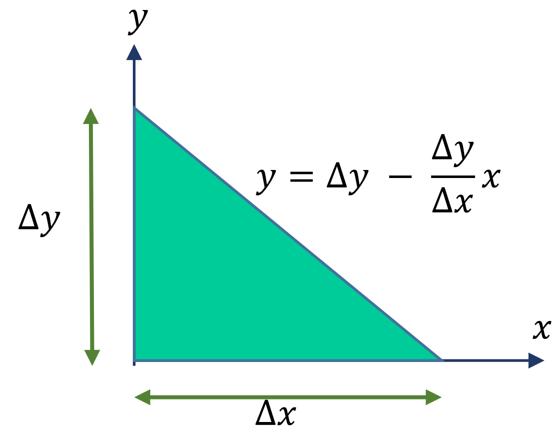
Evaluate

$$\int_S \nabla \cdot \vec{F}\, dx\, dy$$

where



Method 1: $$\iint_S \nabla \cdot \vec{F}\, dx\, dy = \oint_{\partial S} \vec{F} \cdot \hat{n}\, dl$$



$$\nabla \cdot \vec{F}(u) = 2x + 2y$$

$$\int_{x=0}^{\Delta x} \int_{y=0}^{\Delta y - \frac{\Delta y}{\Delta x} x} (2x + 2y)\, dy\ dx$$

$$= \int_{x=0}^{\Delta x} \left[2xy + y^2\right]_{y=0}^{\Delta y - \frac{\Delta y}{\Delta x} x} dx$$

$$= \int_{x=0}^{\Delta x} \left[2x\left(\Delta y - \frac{\Delta y}{\Delta x} x\right) + \left(\Delta y - \frac{\Delta y}{\Delta x} x\right)^2\right] dx$$

$$= \int_{x=0}^{\Delta x} \left[2x\left(\Delta y - \frac{\Delta y}{\Delta x} x\right) + \left(\Delta y - \frac{\Delta y}{\Delta x} x\right)^2\right] dx$$

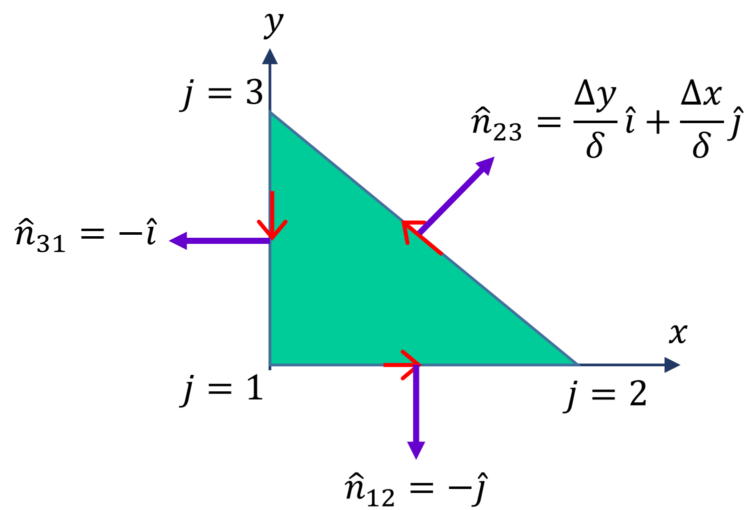
$$= \int_{x=0}^{\Delta x} \left[\Delta y^2 + \left(2\Delta y - 2\frac{\Delta y^2}{\Delta x}\right) x + \left(\frac{\Delta y^2}{\Delta x^2} - 2\frac{\Delta y}{\Delta x}\right) x^2\right] dx$$

$$= \left[x\Delta y^2 + \left(\Delta y - \frac{\Delta y^2}{\Delta x}\right) x^2 + \frac{1}{3}\left(\frac{\Delta y^2}{\Delta x^2} - 2\frac{\Delta y}{\Delta x}\right) x^3\right]_{x=0}^{\Delta x} = \frac{1}{3}\Delta y^2 \Delta x + \frac{1}{3}\Delta y \Delta x^2$$

Method 1: $$\iint_S \nabla \cdot \vec{F}\, dx\, dy = \oint_{\partial S} \vec{F} \cdot \hat{n}\, dl$$

$$\vec{F}(\mathrm{u}) = (x^2 + y^2)\hat{\imath} + (x^2 + y^2)\hat{\jmath}$$

$$= \int_{l_{23}} \vec{F} \cdot \hat{n}_{23}\, dl + \int_{l_{31}} \vec{F} \cdot \hat{n}_{31}\, dl + \int_{l_{12}} \vec{F} \cdot \hat{n}_{12}\, dl$$



To perform the line integral, curves have to be expressed in parametric forms:

$$l_{23} \rightarrow \vec{r}(t) = \underbrace{(1 - t)\,\Delta x}_{x}\, \hat{\imath} + \underbrace{t\,\Delta y}_{y}\, \hat{\jmath} \qquad 0 \le t \le 1$$

$$l_{31} \rightarrow \vec{r}(t) = \underbrace{0}_{x}\, \hat{\imath} + \underbrace{(1 - t)\,\Delta y}_{y}\, \hat{\jmath} \qquad 0 \le t \le 1$$

$$l_{12} \rightarrow \vec{r}(t) = \underbrace{t\,\Delta x}_{x}\, \hat{\imath} + \underbrace{0}_{y}\, \hat{\jmath} \qquad 0 \le t \le 1$$

using the line integral of parametric equations:

Say $\vec{F} \cdot \hat{n} = f(x(t), y(t))$ $\qquad \vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$

$$\int_{\vec{r}(t)} \vec{F} \cdot \hat{n}\; dl = \int_{t=0}^{t=1} f(x(t), y(t))\; |\vec{v}(t)|\, dt \qquad |\vec{v}(t)| = \sqrt{\left(\frac{\partial x(t)}{\partial t}\right)^2 + \left(\frac{\partial y(t)}{\partial t}\right)^2}$$

$$= \int_{l_{23}} \frac{\Delta y}{\delta}(x^2+y^2) + \frac{\Delta x}{\delta}(x^2+y^2)\, dl - \int_{l_{31}} (x^2+y^2)\, dl - \int_{l_{12}} (x^2+y^2)\, dl$$

$$\int_{t=0}^{t=1} \left[\frac{\Delta y}{\delta}(x^2+y^2) + \frac{\Delta x}{\delta}(x^2+y^2)\right] \delta\, dt$$

$$= (\Delta x + \Delta y) \int_{t=0}^{t=1} \left[\left((1-t)\,\Delta x\right)^2 + (t\,\Delta y)^2\right] dt$$

$$= \frac{1}{3}(\Delta x^3 + \Delta x\, \Delta y^2 + \Delta y\, \Delta x^2 + \Delta y^3)$$

$$-\Delta y \int_{t=0}^{t=1} (1-t)^2\, \Delta y^2\, dt$$

$$= -\frac{1}{3}\Delta y^3$$

$$-\Delta x \int_{t=0}^{t=1} t^2\, \Delta x^2\, dt$$

$$= -\frac{1}{3}\Delta x^3$$

$$\therefore \int_{l_{23}} \vec{F}\cdot\hat{n}_{23}\, dl + \int_{l_{31}} \vec{F}\cdot\hat{n}_{31}\, dl + \int_{l_{12}} \vec{F}\cdot\hat{n}_{12}\, dl = \frac{1}{3}\Delta y^2 \Delta x + \frac{1}{3}\Delta y \Delta x^2$$

This result agrees with the calculation by doing the volume integration of the flux divergence directly.

Numerical Integration

To connect the application of Green's theorem in RD method

Numerical Integration

Steady State:

The conditions for *second-order-accurate* are:

1) The integration of the flux-divergence must be third-order-accurate.

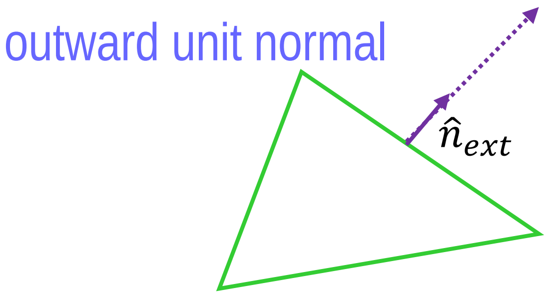
$$\iint_S \nabla \cdot \vec{F}\, dx\, dy = \oint_{\partial S} \vec{F} \cdot \hat{n}_{ext}\, dl = -\oint_{\partial S} \vec{F} \cdot d\vec{n} = 0 + O(h^3)$$

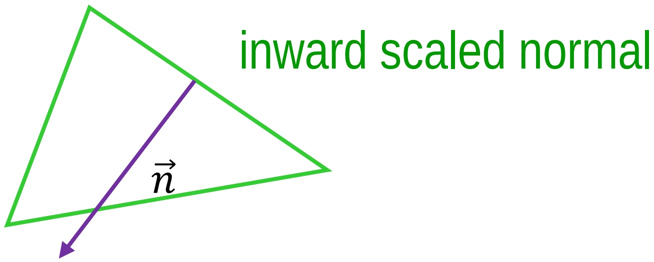
2) The distribution coefficient must be bounded.

$$0 \leq \beta_i^T \leq 1 \Longrightarrow \text{Linear-preserving scheme}$$

eg. LDA

Remarks:





The conditions for *third-order-accurate* are:

1) The integration of the flux-divergence must be fourth-order-accurate.

$$\iint_S \nabla \cdot \vec{F}\, dx\, dy = \oint_{\partial S} \vec{F} \cdot \hat{n}_{ext}\, dl = -\oint_{\partial S} \vec{F} \cdot d\vec{n} = 0 + O(h^4)$$

2) The distribution coefficient must be bounded.

$$0 \le \beta_i^T \le 1$$

INTEGRATION OF FLUX-DIVERGENCE

Integrate the flux crossing the 3 edges

In RD method, flux residual is defined as

$$\phi^T = \iint_S \nabla \cdot \vec{F}\, dx\, dy = \oint_{\partial S} \vec{F} \cdot \hat{n}_{ext}\, dl = -\oint_{\partial S} \vec{F} \cdot d\vec{n}$$

Trapezoidal Rule *Second-Order-Accurate*

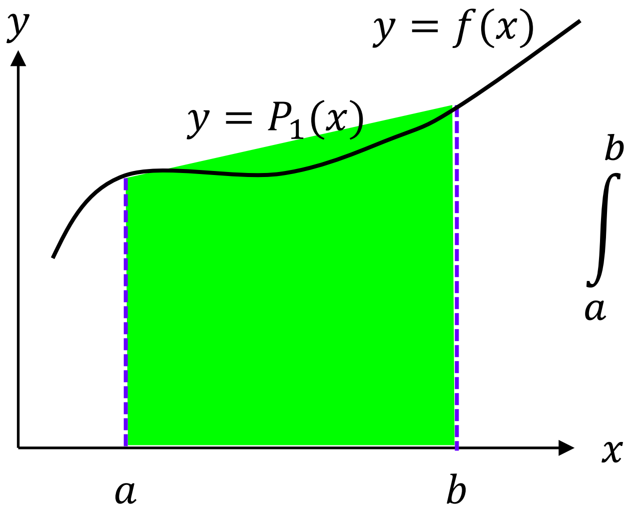
$$\phi^T = +\sum_{j \in T} k_j u_j$$

Simpson's Rule *Third-Order-Accurate*

$$\phi^T = -\sum_{j \in T} \frac{\vec{\lambda} \cdot \vec{n}_j}{6} \left[u_m + 4\, u_{mid,j} + u_p\right]$$

Trapezoidal Rule

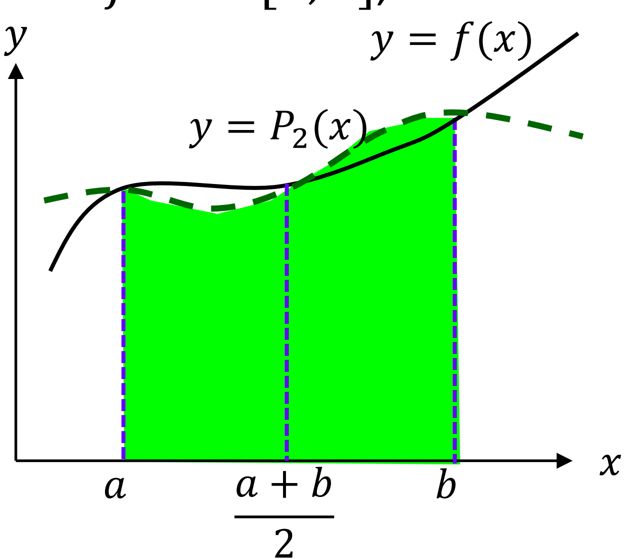
If $f \in C^2[a, b]$, then a number ξ in $[a, b]$ exists with



$$\int_a^b f(x)\,dx = \frac{f(a) + f(b)}{2}(b - a) - \frac{f''(\xi)}{12}(b - a)^3$$

Simpson's Rule

If $f \in C^4[a, b]$, then a number ξ in $[a, b]$ exists and

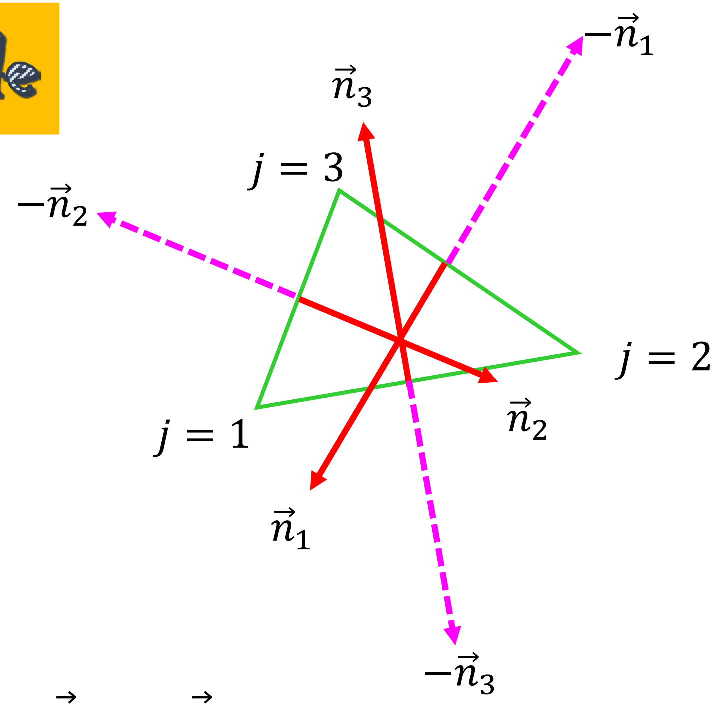


$$\int_a^b f(x)\,dx = \frac{(b - a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{f^{(4)}(\xi)}{2880}(b - a)^5$$

Flux Residual using Trapezoidal Rule

$$\phi^T = +\sum_{j \in T} k_j u_j$$

$$\vec{F}(u) = u\,\vec{\lambda}$$



$$\iint_S \nabla \cdot \vec{F}\, dx\, dy = \oint_{\partial S} \vec{F} \cdot \hat{n}_{ext}\, dl$$

$$= \frac{u_2\vec{\lambda} + u_1\vec{\lambda}}{2} \cdot (-\vec{n}_3) + \frac{u_3\vec{\lambda} + u_2\vec{\lambda}}{2} \cdot (-\vec{n}_1) + \frac{u_1\vec{\lambda} + u_3\vec{\lambda}}{2} \cdot (-\vec{n}_2)$$

$$= \frac{u_1\vec{\lambda}}{2} \cdot (-\vec{n}_2 - \vec{n}_3) + \frac{u_2\vec{\lambda}}{2} \cdot (-\vec{n}_1 - \vec{n}_3) + \frac{u_3\vec{\lambda}}{2} \cdot (-\vec{n}_1 - \vec{n}_2)$$

$$= \frac{u_1\vec{\lambda}}{2} \cdot \vec{n}_1 + \frac{u_2\vec{\lambda}}{2} \cdot \vec{n}_2 + \frac{u_3\vec{\lambda}}{2} \cdot \vec{n}_3$$

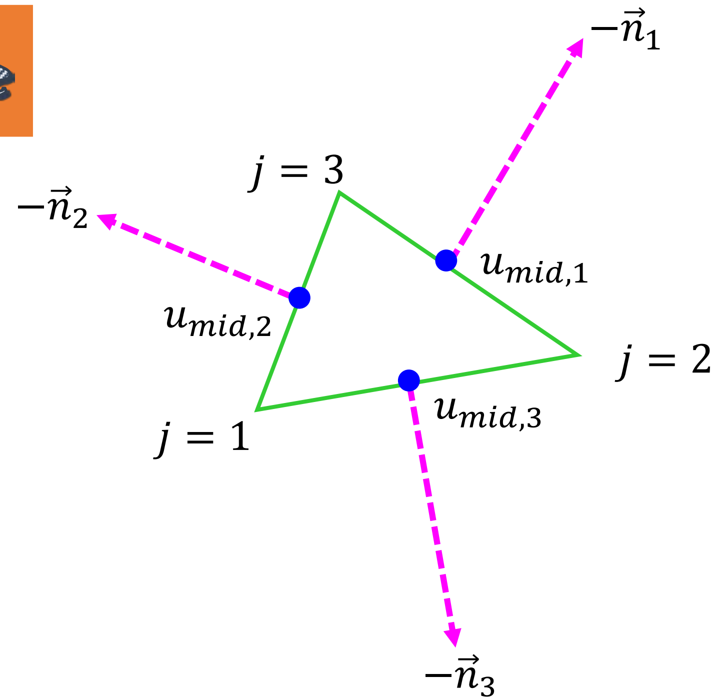
$$= k_1 u_1 + k_2 u_2 + k_3 u_3$$

where $k_j = \dfrac{\vec{\lambda} \cdot \vec{n}_2}{2}$

Flux Residual using Simpson's Rule

$$\phi^T = -\sum_{j\in T} \frac{\vec{\lambda}\cdot\vec{n}_j}{6}\left[u_m + 4\,u_{mid,j} + u_p\right]$$

$$\vec{F}(u) = u\,\vec{\lambda}$$



$$\iint_S \nabla\cdot\vec{F}\,dx\,dy = \oint_{\partial S} \vec{F}\cdot\,\hat{n}_{ext}\,dl$$

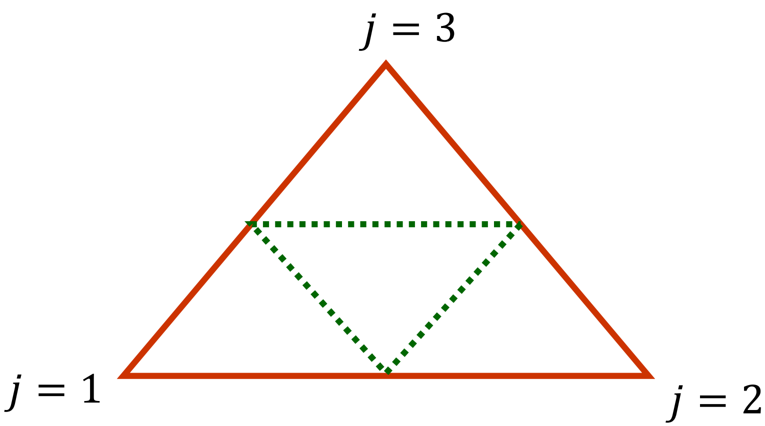
$$= \frac{1}{6}\left[u_1\vec{\lambda} + 4\,u_{mid,3} + u_2\vec{\lambda}\right]\cdot(-\vec{n}_3) + \frac{1}{6}\left[u_2\vec{\lambda} + 4\,u_{mid,1} + u_3\vec{\lambda}\right]\cdot(-\vec{n}_1)$$

$$+\frac{1}{6}\left[u_1\vec{\lambda} + 4\,u_{mid,1} + u_3\vec{\lambda}\right]\cdot(-\vec{n}_2)$$

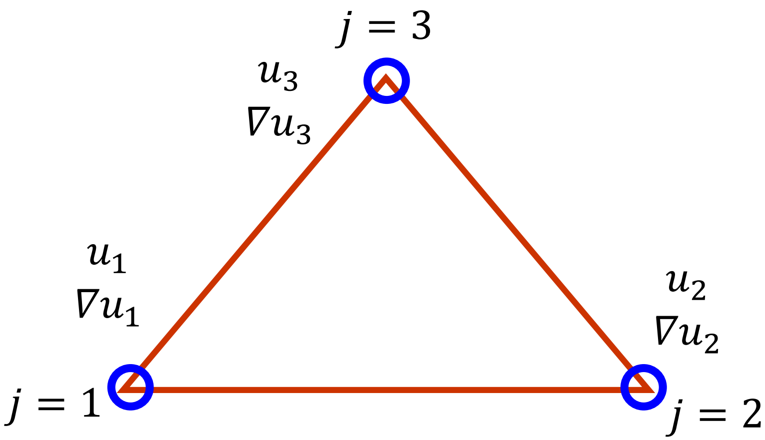
Third-Order-Accurate : Steady

To find the $u_{mid,j}$ is always a concern. There are two ways of doing it:

1) Submesh Reconstruction
2) Gradient Recovery



Submesh Reconstruction

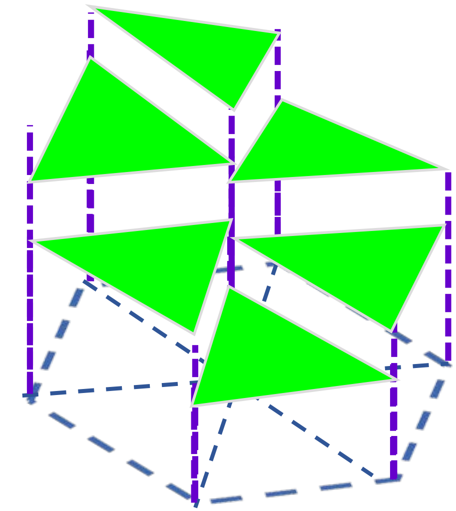


Gradient Recovery

Gradient Recovery

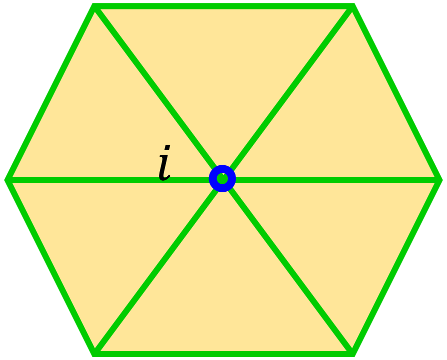
Step 1 : Find the gradient for each cell using Green-Gauss theorem. The gradient for each cell is always a constant

$$\nabla u^T = \frac{1}{S_T} \oint_{\partial T} u\vec{n} \cdot d\Gamma = \frac{1}{2S_T} \sum_{j \in T} u\vec{n}_j$$



Step 2 : The nodal gradient at node i is taken as the weighted average of ∇u^T from the cell sharing it.

$$\nabla u_i = \frac{\sum_{T \in \cup \Delta_i} S_T \nabla u^T}{\sum_{T \in \cup \Delta_i} S_T}$$



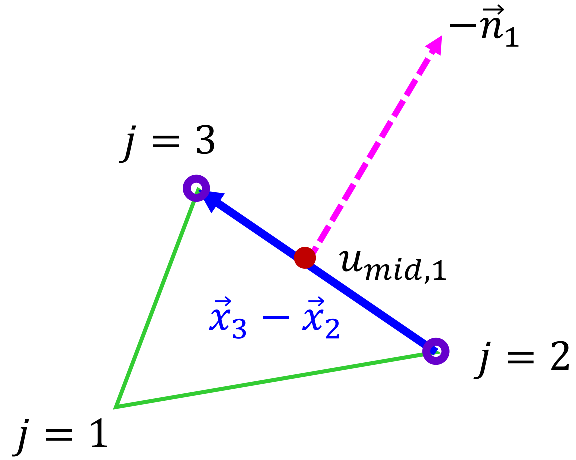
Similar procedures are applied until all the nodal gradient has been evaluated.

Step 3 : Estimating the value of $u_{mid,j}$. Take note that the position of the mid-point, $\vec{x}_m$ is never sought.

$$u_{mid,j} = \frac{u_m + u_p}{2} + \frac{\nabla u_m - \nabla u_p}{8} \cdot (\vec{x}_p - \vec{x}_m)$$

For example, $u_{mid,1}$ is sought as letting

$$m = 2 \qquad p = 3$$



Step 4 : Finally, evaluate the flux residual of a cell using Simpson's rule.

$$\phi^T = -\sum_{j \in T} \frac{\vec{\lambda} \cdot \vec{n}_j}{6} \left[u_m + 4\, u_{mid,j} + u_p \right]$$

Third-Order-Accurate : Unsteady

Aside to the two conditions mentioned for steady case, for unsteady problem, the time derivative has to be third-order-accurate in **time** (i.e. Δt^3) to give the scheme third-order-accurate in **both time and space**.

$$\sum_{T\in\cup\Delta_i}\left\{\underbrace{\sum_{j\in T} m_{ij}^T\frac{\partial u_j(t)}{\partial t}}_{\text{Unsteady residual}}+\underbrace{\beta_i^T\phi^T}_{\text{Steady residual / flux residual}}\right\}=0+O(h^4)$$

For the sake of brevity, let us re-write the unsteady residual as

$$\sum_{T\in\cup\Delta_i}\{\psi_i^T+\phi_i^T\}\cong 0$$

The evaluation of flux residual is already mentioned in previous slides.

The unsteady residual is evaluated as following:

$$\psi_i^T = \left(\beta_j^T - \frac{1}{3}\right) \sum_{j \in T} \frac{S_T}{3} \left(\frac{\partial u}{\partial t}\right)_j + \frac{1}{3} \sum_{j \in T} \frac{S_T}{3} \left(\frac{\partial u}{\partial t}\right)_{mid,j}$$

where

$$\left(\frac{\partial u}{\partial t}\right)_j = \frac{11\, u_j^{n+1} - 18 u_j^n + 9\, u_j^{n-1} - 2 u_j^{n-2}}{6\, \Delta t} + O(\Delta t^3)$$

$$\left(\frac{\partial u}{\partial t}\right)_{mid,j} = \frac{11\, u_{mid,j}^{n+1} - 18 u_{mid,j}^n + 9\, u_{mid,j}^{n-1} - 2 u_{mid,j}^{n-2}}{6\, \Delta t} + O(\Delta t^3)$$

Finally, the equation of system at time level $\boldsymbol{n+1}$ is approximated using dual-time-stepping

$$u_i^{n+1,k+1} = u_i^{n+1,k} - \frac{\Delta \tau}{S_i} \sum_{T \in \cup \Delta_i} \left(\psi_i^T + \phi_i^T\right)^{n+1,k}$$

Numerical Results

Steady Case : Linear & Circular Advection

Unsteady Case : Linear & Circular Advection

Steady Case : Linear Advection

Constant linear advection speed : $\vec{\lambda} = (2,1)^T$

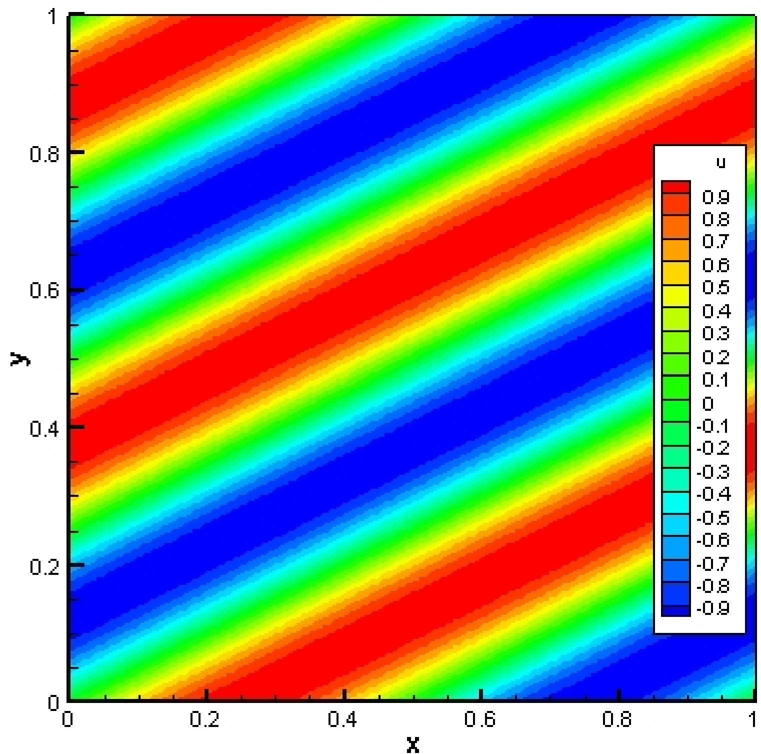
Domain : $\Omega = [0,1]^2 \subseteq \mathbb{R}^2$

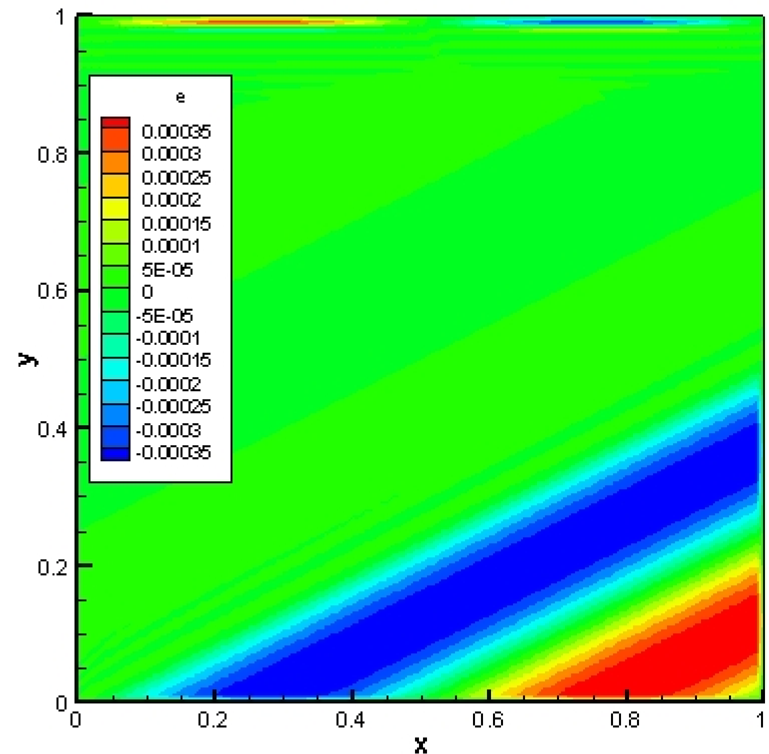
Initial Guess :
$$u(x,y) = \begin{cases} \cos(\pi x) & ; \quad y = 0 \\ 0 & ; \quad 0 < y \leq 1 \end{cases}$$

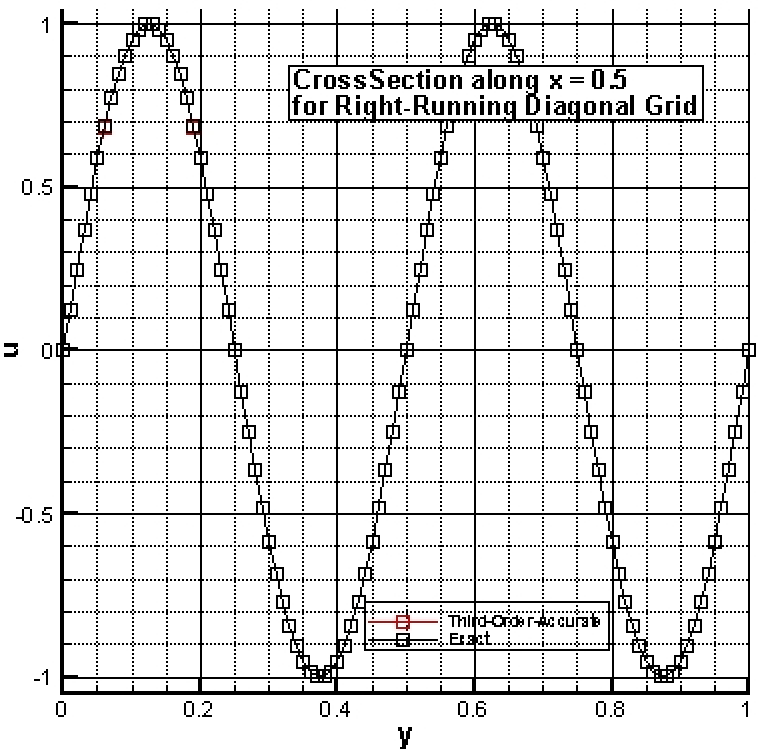
Exact solution (for steady state) : $u(x,y) = \cos[\pi(x - 2\,y)]$

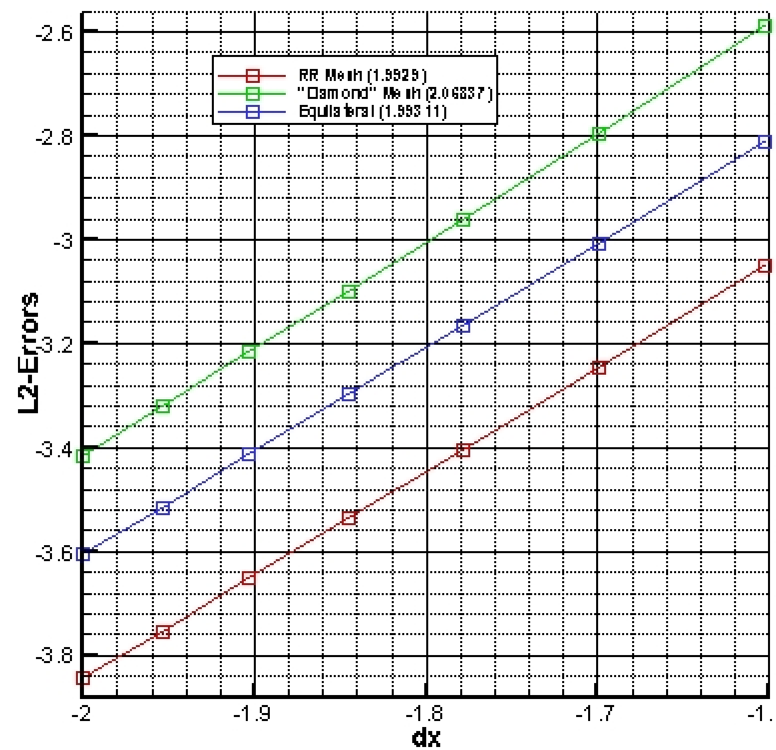
Boundary Conditions : The boundaries are always set to the exact solution.

$$u(x,y) = \cos[\pi(x - 2\,y)] \quad \text{for} \quad (x,y) \in \partial\Omega$$






CrossSection along x = 0.5
for Right-Running Diagonal Grid
Third-Order-Accurate
Exact
u
y
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1


RR Mesh (1.9929)
"Diamond" Mesh (2.06837)
Equilateral (1.99311)
L2-Errors
dx
-2.6
-2.8
-3
-3.2
-3.4
-3.6
-3.8
-2
-1.9
-1.8
-1.7

Steady Case : Circular Advection

Averaged circular velocity : $\vec{\tilde{\lambda}} = (\tilde{y}, -\tilde{x})^T$ where $\tilde{x} = \frac{1}{3}\sum_{j\in T} x_j$ $\tilde{y} = \frac{1}{3}\sum_{j\in T} y_j$

Domain : $\Omega = [-1,1] \times [0,1] \subseteq \mathbb{R}^2$

Initial Guess :

$$u(x,y) = \begin{cases} G(x) & ,\{(x,y): -0.75 \le x \le -0.25, y = 0\} \\ G(-x) & ,\{(x,y): 0.25 \le x \le 0.75, y = 0\} \\ 0 & otherwise \end{cases}$$

in which $G(x) = \begin{cases} g(4x+3) & ,\{(x,y): -0.75 \le x \le -0.5, y = 0\} \\ g(-4x-1) & ,\{(x,y): -0.5 \le x \le -0.25, y = 0\} \end{cases}$

where $g(x) = x^5(70\,x^4 - 315\,x^3 + 540\,x^2 - 420\,x + 126)$

Boundary Conditions :

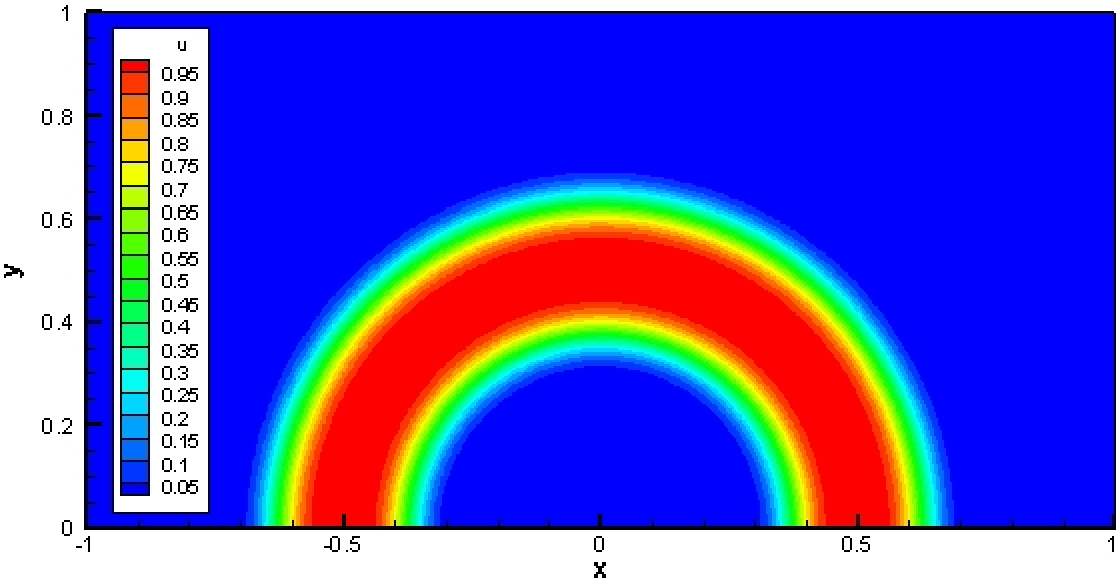
$$u(x,y) = \begin{cases} G(x) & \{(x,y): -0.75 \le x \le -0.25, y = 0\} \\ G(-x) & \{(x,y): 0.25 \le x \le 0.75, y = 0\} \\ 0 & (A \cup B \cup C) \cup \{(x,y): x = -1\} \cup \{(x,y): x = 1\} \cup \{(x,y): y = 1\} \end{cases}$$

where $A = \{(x,y): -1 \le x \le -0.75, y = 0\}$

$B = \{(x,y): -0.25 \le x \le 0.25, y = 0\}$

$C = \{(x,y): 0.75 \le x \le 1, y = 0\}$

Exact solution (for steady state) : $u(x,y) = \begin{cases} G(-r) & , \quad 0.25 \le r \le 0.75 \\ 0 & , \quad elsewhere \end{cases}$ where $r = \sqrt{x^2 + y^2}$


u
0.95
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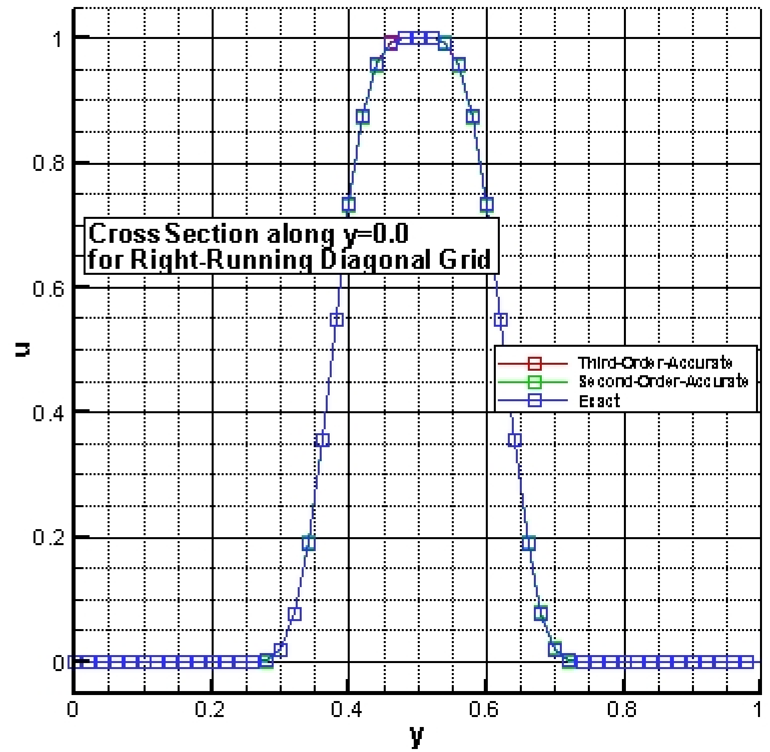
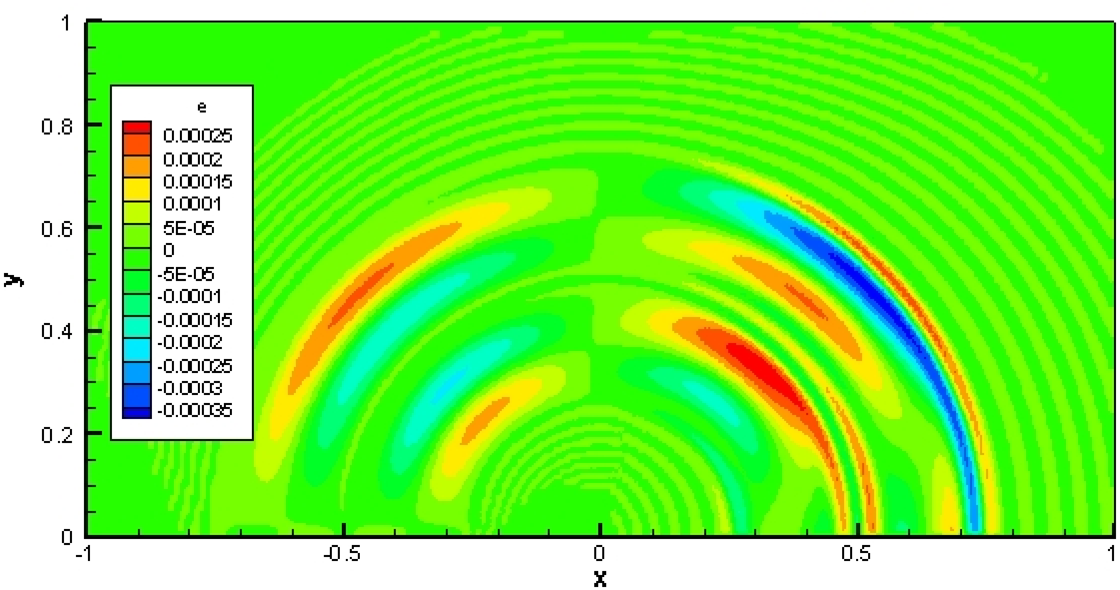
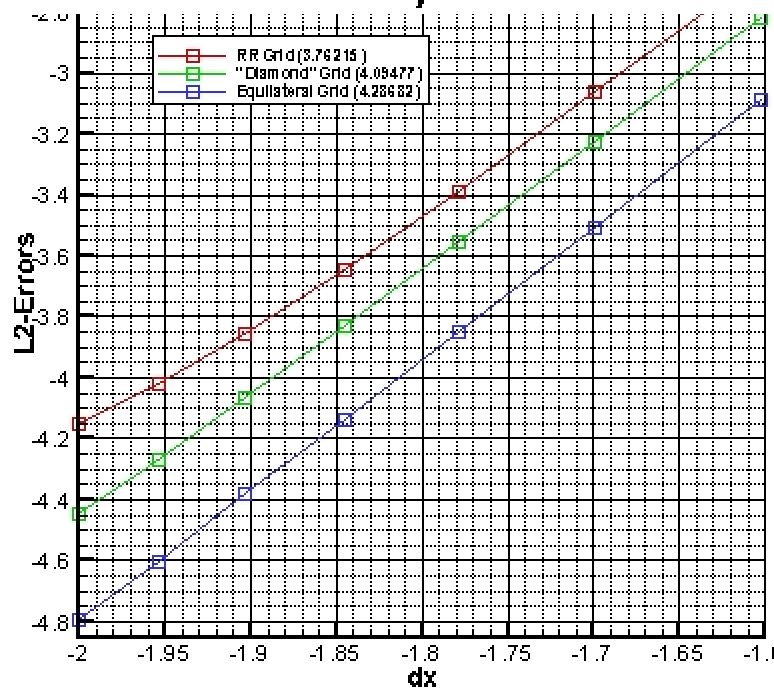
y
x


Cross Section along y=0.0
for Right-Running Diagonal Grid
Third-Order-Accurate
Second-Order-Accurate
Exact
u
y


e
0.00025
0.0002
0.00015
0.0001
5E-05
0
-5E-05
-0.0001
-0.00015
-0.0002
-0.00025
-0.0003
-0.00035
y
x


RR Grid (3.76215)
"Diamond" Grid (4.09477)
Equilateral Grid (4.28682)
L2-Errors
dx

Unsteady Case : Linear Advection

Constant linear advection speed : $\vec{\lambda} = (1,2)^T$

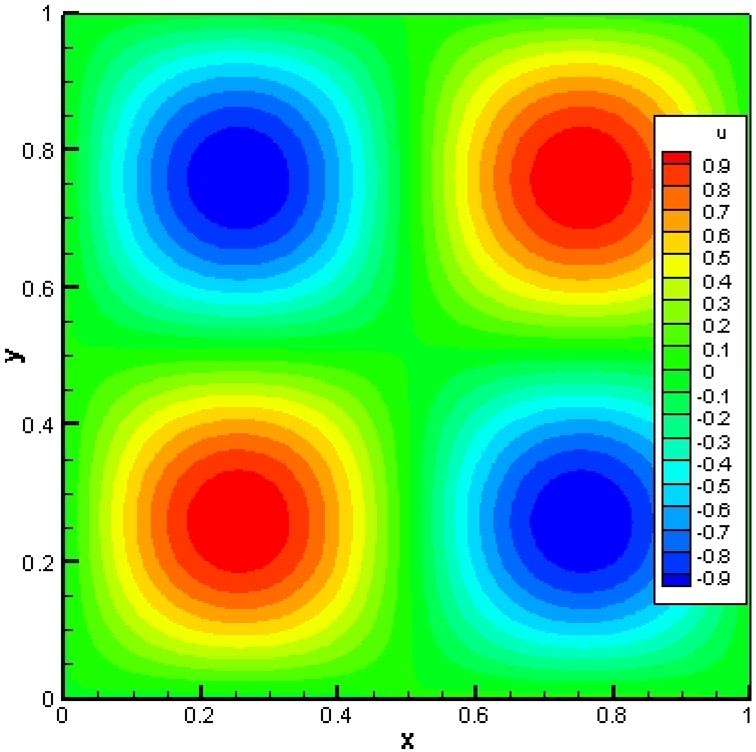
Domain : $\Omega_t = \Omega \times [0, t_f] = [0,1]^2 \times [0,1] \subseteq \mathbb{R}^2 \times \mathbb{R}^+$

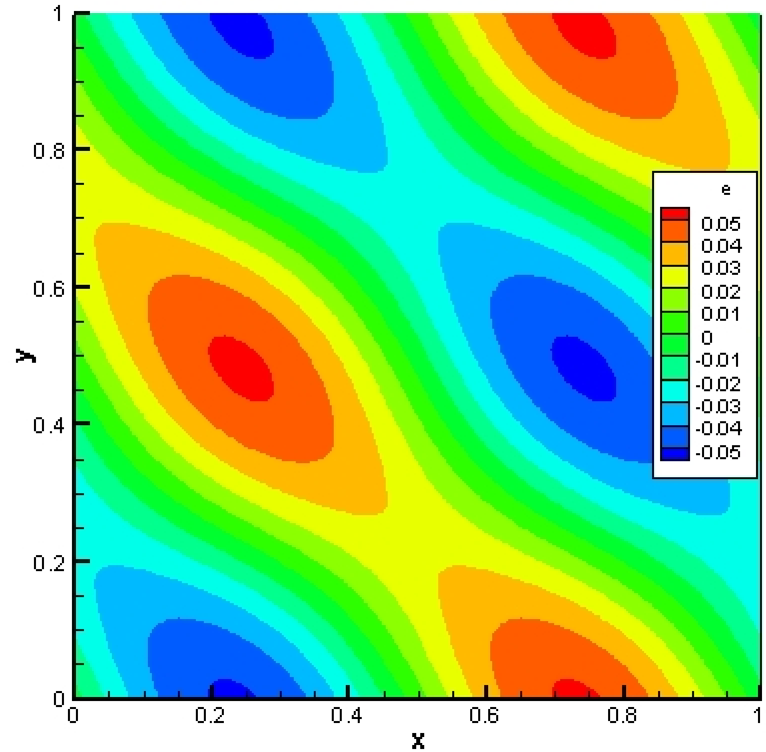
Initial Condition :

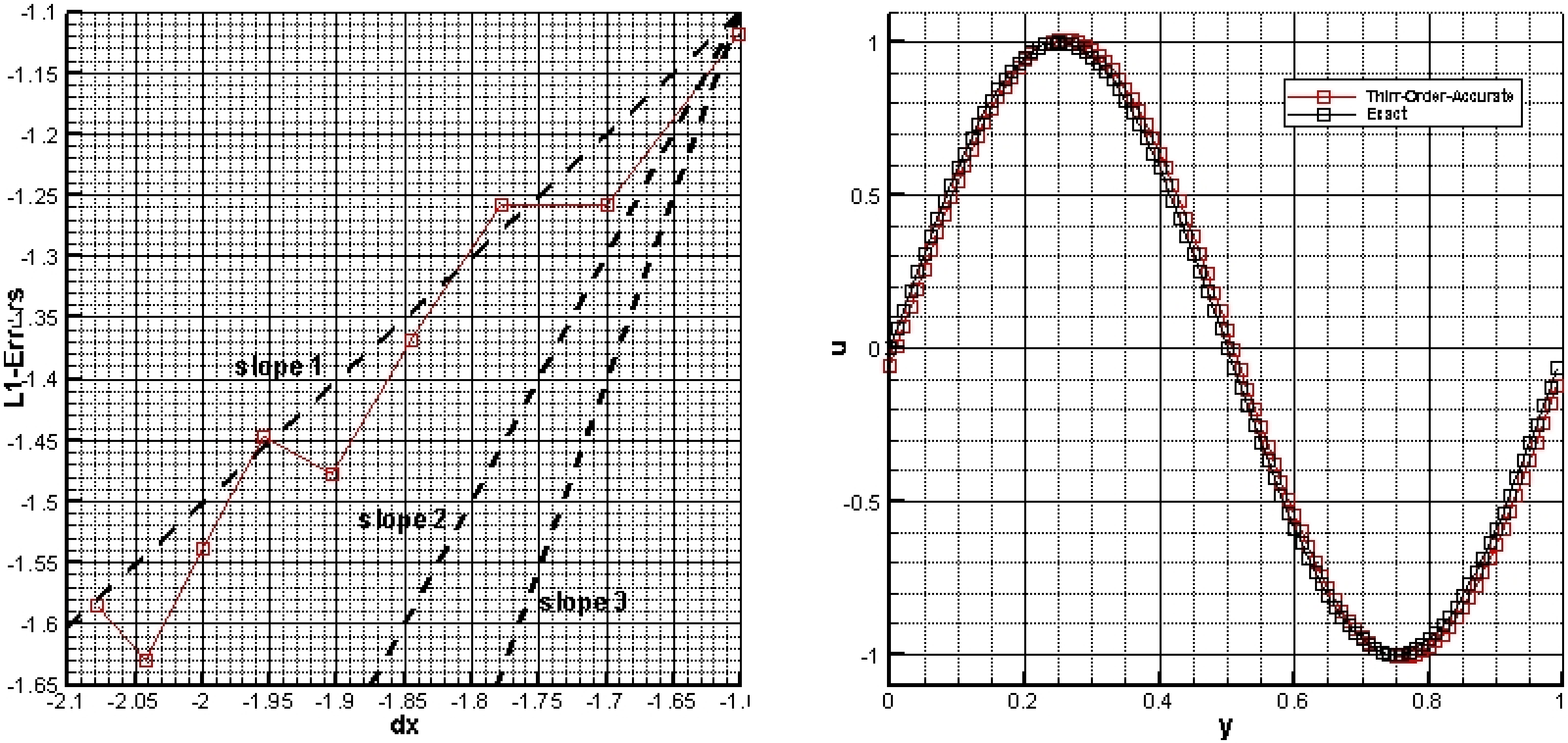
Final solution :

$$u(x,y,0) = u(x,y,1) = \sin(2\pi x)\sin(2\pi y)$$

Boundary Conditions : Periodic boundary condition for $(x,y) \in \partial\Omega$







L1-Errors
dx
slope 1
slope 2
slope 3
u
y
Third-Order-Accurate
Exact

Unsteady Case : Circular Advection

Averaged circular velocity : $\tilde{\vec{\lambda}} = (-2\pi\tilde{y}, 2\pi\tilde{x})^T$ where $\tilde{x} = \frac{1}{3}\sum_{j\in T} x_j$ $\tilde{y} = \frac{1}{3}\sum_{j\in T} y_j$

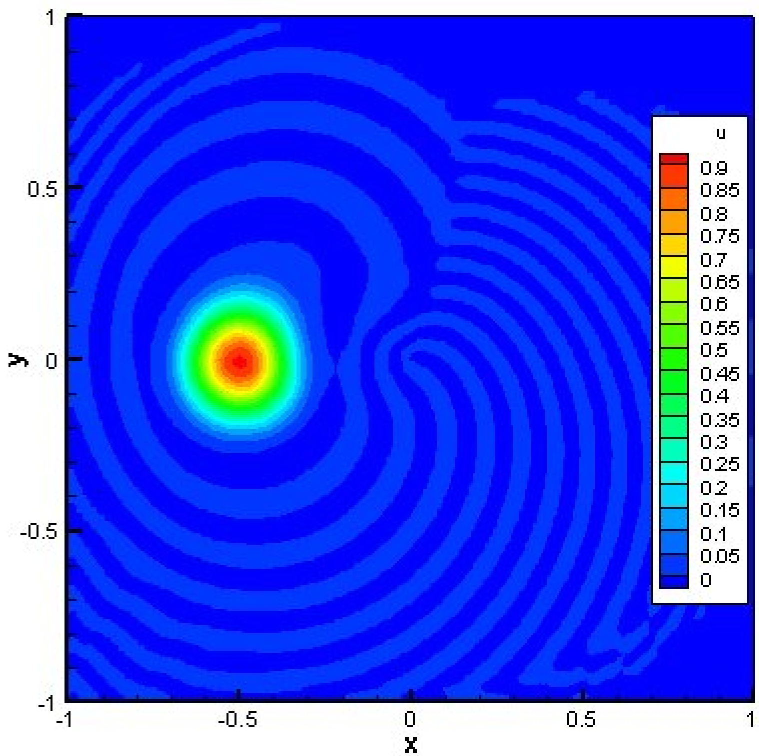
Domain : $\Omega_t = \Omega \times [0, t_f] = [1,1]^2 \times [0,1] \subseteq \mathbb{R}^2 \times \mathbb{R}^+$

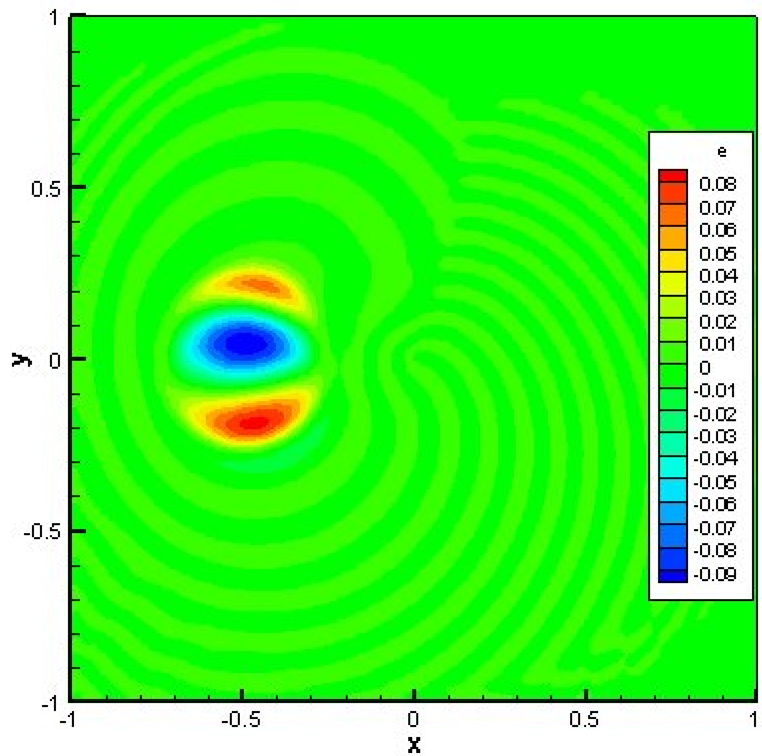
Initial Condition :

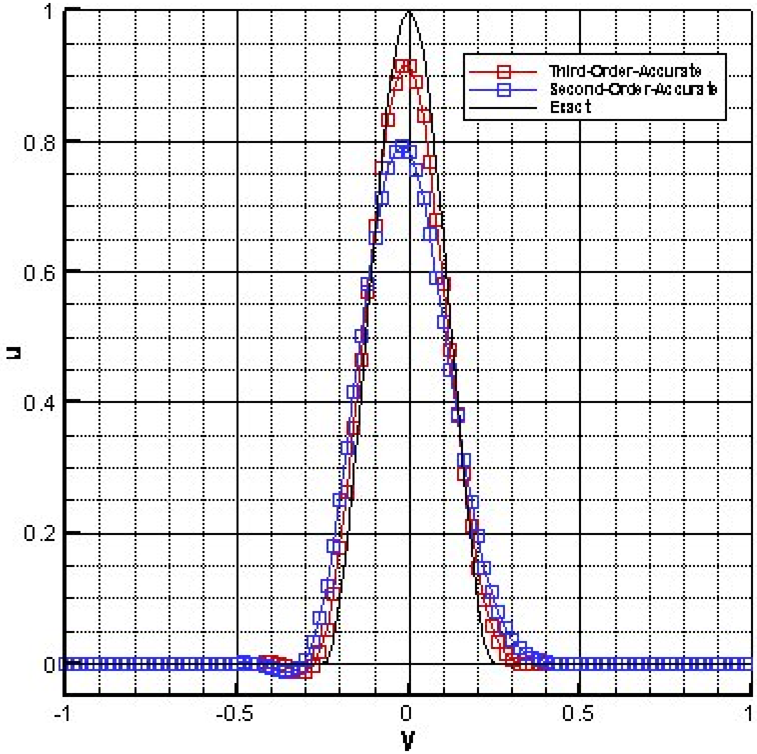
Final solution :

$$u(x,y,0) = u(x,y,1) = \begin{cases} cos^2(2\pi r) & r \leq 0.25 \\ 0 & r > 0.25 \end{cases} \quad \text{where} \quad r = \sqrt{(x+0.5)^2 + y^2}$$

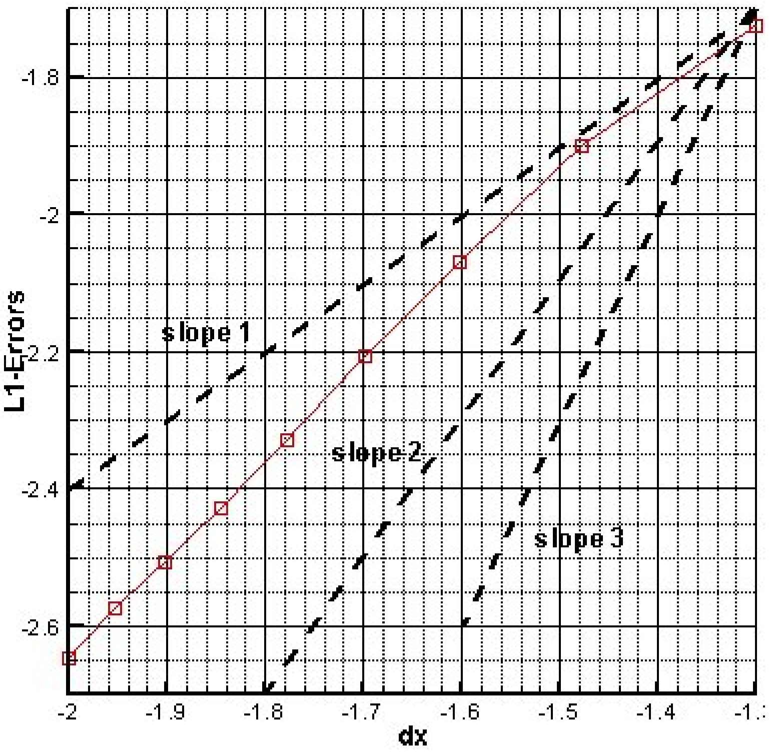
Boundary Conditions : All the boundaries are always set to be 0. $u(x,y,t) = 0$ for $(x,y) \in \partial\Omega$







Third-Order-Accurate
Second-Order-Accurate
Exact
u
y
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1



L1-Errors
dx
slope 1
slope 2
slope 3
-1.8
-2
-2.2
-2.4
-2.6
-2
-1.9
-1.8
-1.7
-1.6
-1.5
-1.4

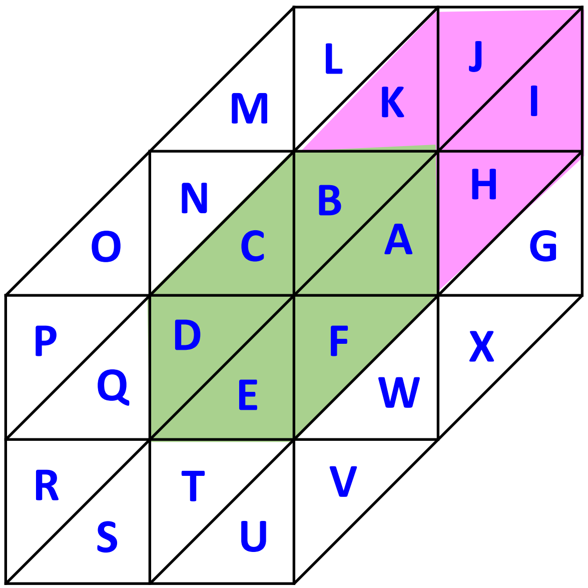
Truncation Errors Analysis

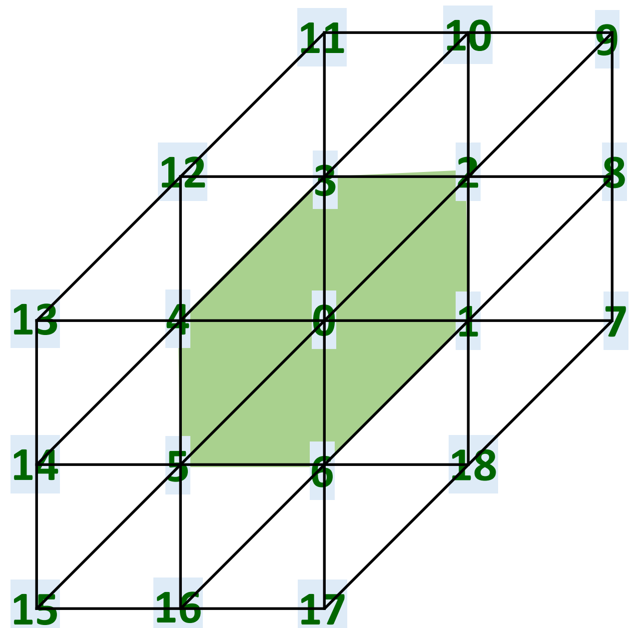
Steady Case : $a_x > a_y$

Unsteady Case : $a_x < a_y$

Truncation Errors : Steady

The disadvantage of using gradient recovery approach in estimating $u_{mid,j}$ is that the scheme is no longer on compact stencil.





Expand the Taylor's expansion series in x and y-coordinates. For examples,

$$u_1 = u_0 + \Delta x \frac{\partial u_0}{\partial x} + \frac{1}{2} \Delta x^2 \frac{\partial^2 u_0}{\partial x^2} + O(\Delta x^3)$$

$$u_8 = u_0 + \left((2\,\Delta x) \frac{\partial u_0}{\partial x} + (\Delta y) \frac{\partial u_0}{\partial y} \right) + \left((2\,\Delta x)^2 \frac{\partial^2 u_0}{\partial x^2} + 2\,(2\,\Delta x)(\Delta y) \frac{\partial^2 u_0}{\partial x \partial y} + (\Delta y)^2 \frac{\partial^2 u_0}{\partial y^2} \right) + O(\Delta x^3, \Delta y^3)$$

$$u_{16} = u_0 + \left((-\,\Delta x) \frac{\partial u_0}{\partial x} + (-\,2\,\Delta y) \frac{\partial u_0}{\partial y} \right) + \left((-\,\Delta x)^2 \frac{\partial^2 u_0}{\partial x^2} + 2\,(-\,\Delta x)(-\,2\,\Delta y) \frac{\partial^2 u_0}{\partial x \partial y} + (-\,2\,\Delta y)^2 \frac{\partial^2 u_0}{\partial y^2} \right) + O(\Delta x^3, \Delta y^3)$$

Step 1 : Evaluate cell gradient using the formula:

$$\nabla u^T = \frac{1}{2S_T} \sum_{j \in T} u \vec{n}_j$$

For examples,

$$\nabla u^A = \left(\frac{u_1 - u_0}{\Delta x}\right) \hat{\imath} + \left(\frac{u_2 - u_1}{\Delta y}\right) \hat{\jmath} \qquad \nabla u^J = \left(\frac{u_9 - u_{10}}{\Delta x}\right) \hat{\imath} + \left(\frac{u_{10} - u_2}{\Delta y}\right) \hat{\jmath}$$

Step 2 : Find the nodal gradient numerically, using the results from step 1.

$$\nabla u_i = \frac{\sum_{T \in \cup \Delta_i} S_T \nabla u^T}{\sum_{T \in \cup \Delta_i} S_T}$$

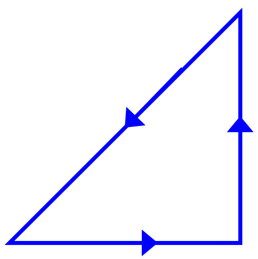
For example,

$$\nabla u_0 = \left(\frac{2\,u_1 + u_2 - u_3 - 2\,u_4 - u_5 + u_6}{\Delta x}\right) \hat{\imath} + \left(\frac{u_2 - u_1 + 2\,u_3 + u_4 - u_5 - 2\,u_6}{\Delta y}\right) \hat{\jmath}$$

Step 3 : Evaluate the flux residual using Simpson's rule.

$$\phi^T = -\sum_{j \in T} \frac{\vec{\lambda} \cdot \vec{n}_j}{6} \left[u_m + 4\, u_{mid,j} + u_p \right] = -\sum_{j \in T} \frac{k_j}{3} \left[u_m + 4\, u_{mid,j} + u_p \right]$$

For triangular cell $T = A$



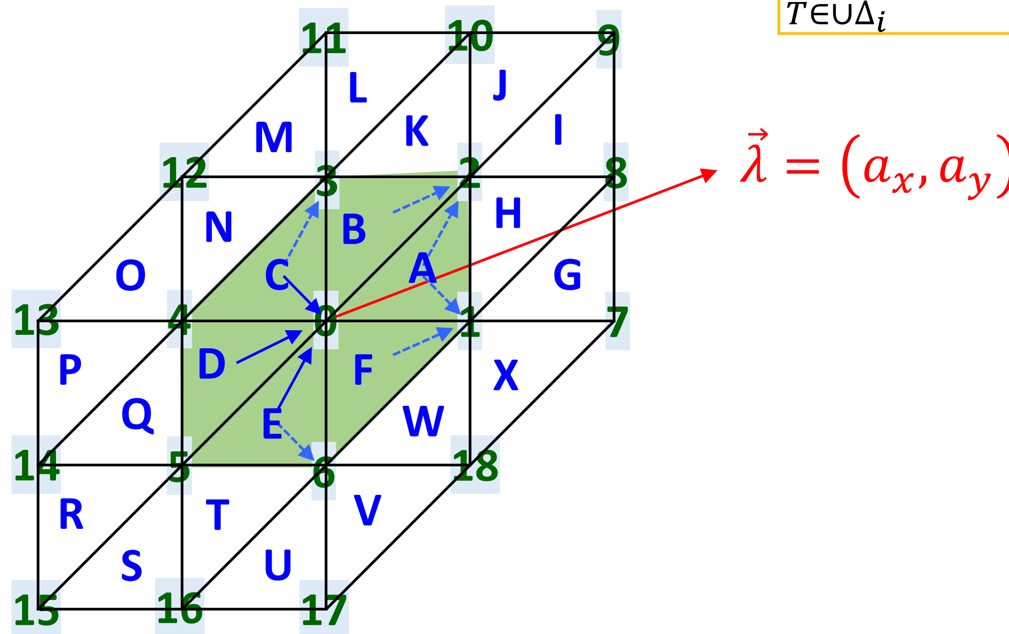
$$\phi^A = -\frac{k_0^A}{3} \left[3\, u_1 + 3\, u_2 - \frac{1}{2}\, \Delta y\, \hat{j} \cdot (\nabla u_2 - \nabla u_1) \right]$$
$$- \frac{k_1^A}{3} \left[3\, u_0 + 3\, u_2 + \frac{1}{2}\, \Delta x\, \hat{\imath} \cdot (\nabla u_0 - \nabla u_2) + \frac{1}{2}\, \Delta y\, \hat{j} \cdot (\nabla u_0 - \nabla u_2) \right]$$
$$- \frac{k_2^A}{3} \left[3\, u_0 + 3\, u_1 - \frac{1}{2}\, \Delta x\, \hat{\imath} \cdot (\nabla u_1 - \nabla u_0) \right]$$

where

$$k_0^A = -\frac{1}{2}\, a_x\, \Delta y$$
$$k_1^A = \frac{1}{2}\, a_x\, \Delta y - \frac{1}{2}\, a_y\, \Delta x$$
$$k_2^A = \frac{1}{2}\, a_y\, \Delta x$$

Step 4 : Assembling the flux residual to node $i = 0$.

$$\sum_{T \in \cup\Delta_i} \{\beta_i^T \phi^T\} \cong 0$$



For linear advection speed $a_x > a_y$, as illustrated in figure above, only element $T = C, D, E$ will contribute to node $i = 0$.

$$\sum_{T \in \cup\Delta_0} \beta_0^T \phi^T = \beta_0^C \phi^C + \beta_0^D \phi^D + \beta_0^D \phi^D$$

From $T = C$

$$\beta_0^C = 1 - \frac{a_y \, \Delta x}{a_x \, \Delta y}$$

$$\phi^C = -\frac{k_0^C}{3}\left[3u_0 + 3u_3 - \frac{1}{2}\Delta y \, \hat{j} \cdot (\nabla u_3 - \nabla u_0)\right] - \frac{k_1^C}{3}\left[3u_3 + 3u_4 + \left(\frac{1}{2}\Delta x \, \hat{\imath} + \frac{1}{2}\Delta y \, \hat{j}\right) \cdot (\nabla u_4 - \nabla u_3)\right] - \frac{k_2^C}{3}\left[3u_0 + 3u_4 - \frac{1}{2}\Delta x \, \hat{\imath} \cdot (\nabla u_0 - \nabla u_4)\right]$$

where $k_0^C = -\frac{1}{2} a_x \, \Delta y$ $\quad k_1^C = \frac{1}{2} a_x \, \Delta y - \frac{1}{2} a_y \, \Delta x$ $\quad k_2^C = \frac{1}{2} a_y \, \Delta x$

From $T = D$

$$\beta_0^D = 1$$

$$\phi^D = -\frac{k_5^D}{3}\left[3u_0 + 3u_4 + \frac{1}{2}\Delta x \, \hat{\imath} \cdot (\nabla u_4 - \nabla u_0)\right] - \frac{k_0^D}{3}\left[3u_4 + 3u_5 + \frac{1}{2}\Delta y \, \hat{j} \cdot (\nabla u_5 - \nabla u_4)\right] - \frac{k_4^D}{3}\left[3u_0 + 3u_5 - \left(\frac{1}{2}\Delta x \, \hat{\imath} + \frac{1}{2}\Delta y \, \hat{j}\right)\hat{\imath} \cdot (\nabla u_0 - \nabla u_5)\right]$$

where $k_5^D = -\frac{1}{2} a_y \, \Delta x$ $\quad k_0^D = \frac{1}{2} a_x \, \Delta y$ $\quad k_4^D = -\frac{1}{2} a_x \, \Delta y + \frac{1}{2} a_y \, \Delta x$

From $T = E$

$$\beta_0^E = \frac{a_y \, \Delta x}{a_x \, \Delta y}$$

$$\phi^E = -\frac{k_5^E}{3}\left[3u_0 + 3u_6 - \frac{1}{2}\Delta y \, \hat{j} \cdot (\nabla u_0 - \nabla u_6)\right] - \frac{k_6^E}{3}\left[3u_0 + 3u_5 + \left(\frac{1}{2}\Delta x \, \hat{\imath} + \frac{1}{2}\Delta y \, \hat{j}\right) \cdot (\nabla u_5 - \nabla u_0)\right] - \frac{k_0^E}{3}\left[3u_5 + 3u_6 - \frac{1}{2}\Delta x \, \hat{\imath} \cdot (\nabla u_5 - \nabla u_6)\right]$$

where $k_5^E = -\frac{1}{2} a_x \, \Delta y$ $\quad k_6^E = \frac{1}{2} a_x \, \Delta y - \frac{1}{2} a_y \, \Delta x$ $\quad k_0^E = \frac{1}{2} a_y \, \Delta x$

Truncation Errors : Unsteady

The flux residual at time level $n + 1$ is evaluated exactly the same as for the steady case.

However, when it comes to the unsteady residual, time has to be considered as the third variable.

$$u(t^{n+1} \pm \Delta t, x_0 \pm \Delta x, y_0 \pm \Delta y)$$

$$= u_0^{n+1} + \left((\pm\Delta t) \frac{\partial}{\partial t} + (\pm\Delta x) \frac{\partial}{\partial x} + (\pm\Delta y) \frac{\partial}{\partial y} \right) u_0^{n+1}$$

$$+ \frac{1}{2}\left((\pm\Delta t)^2 \frac{\partial^2}{\partial t^2} + (\pm\Delta x)^2 \frac{\partial^2}{\partial x^2} + (\pm\Delta y)^2 \frac{\partial^2}{\partial y^2} + 2(\pm\Delta t)(\pm\Delta x) \frac{\partial^2}{\partial t \partial x} + 2(\pm\Delta t)(\pm\Delta y) \frac{\partial^2}{\partial t \partial y}\right.$$

Steps employed in approximating $u_{mid,j}$ have to be repeated here, but with the Taylor's expansion series including the time, $u(t^{n+1} \pm \Delta t, x_0 \pm \Delta x, y_0 \pm \Delta y)$

The contribution of unsteady residual from each element to node $i = 0$ has to be considered.

$$\sum_{T \in \cup \Delta_i} \{\psi_i^T + \phi_i^T\} \cong 0$$

where the unsteady residual generally involves discretization of time derivative to third-order-accurate in time.

$$\left(\frac{\partial u}{\partial t}\right)_j = \frac{11\, u_j^{n+1} - 18 u_j^n + 9\, u_j^{n-1} - 2 u_j^{n-2}}{6\, \Delta t} + O(\Delta t^3)$$

$$\left(\frac{\partial u}{\partial t}\right)_{mid,j} = \frac{11\, u_{mid,j}^{n+1} - 18 u_{mid,j}^n + 9\, u_{mid,j}^{n-1} - 2 u_{mid,j}^{n-2}}{6\, \Delta t} + O(\Delta t^3)$$

TE analysis for Steady Case

linear advection speed $a_x > a_y$

$$\sum_{T\in\cup\Delta_i}(\beta_i^T\phi_i^T) = \Delta x\,\Delta y\left(a_x\frac{\partial}{\partial x}+a_y\frac{\partial}{\partial y}\right)u_0 - \frac{1}{2}\frac{\Delta x^2\,\Delta y}{a_x}\left({a_x}^2\frac{\partial^2}{\partial x^2}+{a_y}^2\frac{\partial^2}{\partial y^2}+2a_xa_y\frac{\partial^2}{\partial x\partial y}\right)u_0 + O(\Delta x^3,\Delta y^3) \qquad \text{--(a)}$$

From the governing equation,

$$a_x\frac{\partial u}{\partial x}+a_y\frac{\partial u}{\partial y}=0 \qquad \text{--(b)}$$

Multiply by a_x and differentiate w.r.t x

Multiply by a_y and differentiate w.r.t y

$$a_x\frac{\partial}{\partial x}\left(a_x\frac{\partial u}{\partial x}+a_y\frac{\partial u}{\partial y}\right)=a_x\frac{\partial}{\partial x}(0) \quad + \quad a_y\frac{\partial}{\partial y}\left(a_x\frac{\partial u}{\partial x}+a_y\frac{\partial u}{\partial y}\right)=a_y\frac{\partial}{\partial y}(0)$$

$${a_x}^2\frac{\partial^2 u}{\partial x^2}+2\,a_xa_y\frac{\partial^2 u}{\partial x\partial y}+{a_y}^2\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{--(c)}$$

Substitute equations (b) and (c) into (a),

$$\sum_{T\in\cup\Delta_i}(\beta_i^T\phi_i^T) = O(\Delta x^3,\Delta y^3)$$

TE analysis for Unsteady Case

linear advection speed $a_x < a_y$

$$\sum_{T\in\cup\Delta_i}\left(\psi_i^T+\phi_i^T\right)^{n+1}=\Delta x\,\Delta y\left(-\frac{1}{3}\frac{\partial}{\partial t}+a_x\frac{\partial}{\partial x}+a_y\frac{\partial}{\partial y}\right)u_0^{n+1}$$

$$-\frac{1}{2}\frac{\Delta x\Delta y^2}{a_y}\left(0\frac{\partial^2}{\partial t^2}+a_x{}^2\frac{\partial^2}{\partial x^2}+a_y{}^2\frac{\partial^2}{\partial y^2}+a_x\frac{\partial^2}{\partial t\partial x}+a_y\frac{\partial^2}{\partial t\partial y}+2a_xa_y\frac{\partial^2}{\partial x\partial y}\right)u_0^{n+1}+O(\Delta t^3,\Delta x^3,\Delta y^3)$$

--(a)

From the governing equation,

$$\frac{\partial u}{\partial t}+a_x\frac{\partial u}{\partial x}+a_y\frac{\partial u}{\partial y}=0 \qquad \text{--(b)}$$

$$\frac{\partial u}{\partial t}=-a_x\frac{\partial u}{\partial x}-a_y\frac{\partial u}{\partial y}$$

$$\left(\frac{\partial}{\partial t}\right)u=\left(-a_x\frac{\partial}{\partial x}-a_y\frac{\partial}{\partial y}\right)u$$

Apply this operator on $\frac{\partial u}{\partial x}$

$$\left(\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial x}=\left(-a_x\frac{\partial}{\partial x}-a_y\frac{\partial}{\partial y}\right)\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 u}{\partial t\partial x}=-a_x\frac{\partial^2 u}{\partial x^2}-a_y\frac{\partial^2 u}{\partial y\partial x}$$

$$a_x\frac{\partial^2 u}{\partial t\partial x}=-a_x{}^2\frac{\partial^2 u}{\partial x^2}-a_xa_y\frac{\partial^2 u}{\partial y\partial x}$$

Apply this operator on $\frac{\partial u}{\partial y}$

$$\left(\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial y}=\left(-a_x\frac{\partial}{\partial x}-a_y\frac{\partial}{\partial y}\right)\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial t\partial y}=-a_x\frac{\partial^2 u}{\partial x\partial y}-a_y\frac{\partial^2 u}{\partial y^2}$$

$$a_y\frac{\partial^2 u}{\partial t\partial y}=-a_xa_y\frac{\partial^2 u}{\partial x\partial y}-a_y{}^2\frac{\partial^2 u}{\partial y^2}$$

+

$$a_x\frac{\partial^2 u}{\partial t\partial x}+a_y\frac{\partial^2 u}{\partial t\partial y}=-a_x{}^2\frac{\partial^2 u}{\partial x^2}-2a_xa_y\frac{\partial^2 u}{\partial y\partial x}-a_y{}^2\frac{\partial^2 u}{\partial y^2} \qquad \text{--(c)}$$

In the last expression, we have used the property of

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

which holds if and only if $u(x, y)$ is *smooth* and *continuous*.

Substitute equations (b) and (c) into (a), one yields

$$\sum_{T \in \cup \Delta_i} \left(\psi_i^T + \phi_i^T\right)^{n+1} = -\frac{4\Delta x\ \Delta y}{3} \frac{\partial u_0^{n+1}}{\partial t} + O(\Delta t^3, \Delta x^3, \Delta y^3)$$

The appearance of the $-\frac{4\Delta x\ \Delta y}{3} \frac{\partial u_0^{n+1}}{\partial t}$ term is the main reason on why the method fails to behave third-order-accurate in space for unsteady problem.