Unsteady Scalar Conservation Law: Implicit & Explicit

Group of Computational Fluid Dynamics
School of Aerospace Engineering
Universiti Sains Malaysia
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Presenter : Neoh Soon Sien
Supervisor : Dr. Farzad Ismail

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Implicit Five-Stage Runge-Kutta Method

- In this section, we will discuss about the implicit five-stage Runge-Kutta method. (A Jameson 1991)
- Next, we will investigate the implicit scheme with pseudo-time iteration without the five-stage Runge-Kutta subiterations. (Rossiello et al.)

Implicit Five-Stage Runge-Kutta Method

This method is originally proposed by A. Jameson in year 1991.
Scalar conservation law is given by

$$\frac{\partial u}{\partial t} + \vec{a} \cdot \nabla u = 0$$

In **linear advection** problem, $\vec{a}$ is always constant for all time step.

Two time-level

$$\sum_{T\in\Delta_i} \sum_{j\in T} m_{ij}^T \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2} \sum_{T\in\Delta_i} \beta_i^T (\phi^{T,n+1} + \phi^{T,n}) = 0$$

$$\phi^{T,n+\frac{1}{2}} = \frac{1}{2}(\phi^{T,n+1} + \phi^{T,n})$$

Three time-level

$$\sum_{T\in\Delta_i} \sum_{j\in T} m_{ij}^T \frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} + \sum_{T\in\Delta_i} \beta_i^T \, \phi^{T,n+1} = 0$$

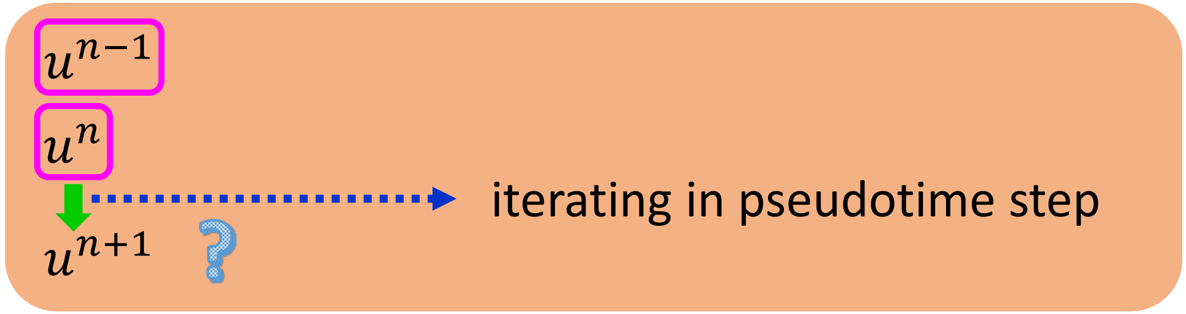
In **non-linear** advection problem, $\vec{a}$ changes with time.

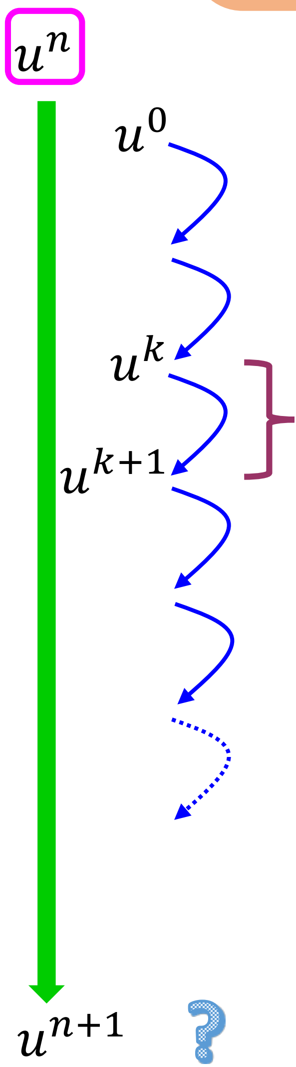
Two time-level

$$\sum_{T\in\Delta_i} \sum_{j\in T} \left(m_{ij}^{T,n+1} \frac{u_j^{n+1}}{\Delta t} - m_{ij}^{T,n} \frac{u_j^n}{\Delta t} \right) + \sum_{T\in\Delta_i} \left(\frac{1}{2} \beta_i^{T,n+1} \phi^{T,n+1} + \frac{1}{2} \beta_i^{T,n} \phi^{T,n} \right) = 0$$

Three time-level

$$\sum_{T\in\Delta_i} \sum_{j\in T} \left(m_{ij}^{T,n+1} \frac{3u_j^{n+1}}{2\Delta t} - m_{ij}^{T,n} \frac{2u_j^n}{\Delta t} + m_{ij}^{T,n-1} \frac{u_j^{n-1}}{2\Delta t} \right) + \sum_{T\in\Delta_i} \beta_i^{T,n+1} \phi^{T,n+1} = 0$$





$$u^{(0)} = u^{k}$$

$$u^{(1)} = u^{(0)} - \alpha_1 \frac{\Delta t^*}{S_i} R\left(u^{(0)}\right)$$

$$u^{(2)} = u^{(0)} - \alpha_2 \frac{\Delta t^*}{S_i} R\left(u^{(1)}\right)$$

$$u^{(3)} = u^{(0)} - \alpha_3 \frac{\Delta t^*}{S_i} R\left(u^{(2)}\right)$$

$$u^{(4)} = u^{(0)} - \alpha_4 \frac{\Delta t^*}{S_i} R\left(u^{(3)}\right)$$

$$u^{(5)} = u^{(0)} - \alpha_5 \frac{\Delta t^*}{S_i} R\left(u^{(4)}\right)$$

$$u^{k+1} = u^{(5)}$$

Stage coefficients are:

$\alpha_1 = 0.0695$

$\alpha_2 = 0.1602$

$\alpha_3 = 0.2898$

$\alpha_4 = 0.5060$

$\alpha_5 = 1.0000$

Physical and Pseudo-time steps:

$$\Delta t_i^* = \Delta t_i = CFL \, \frac{2}{3} \min_{T \in \Delta_i} \frac{S_T}{\sum_{j \in T} k_j^+}$$

In **linear advection** problem:

Two time-level

$$R(u^{(l)}) = \sum_{T\in\Delta_i} \sum_{j\in T} m_{ij}^T \frac{u_j^{(l)} - u_j^n}{\Delta t} + \frac{1}{2} \sum_{T\in\Delta_i} \beta_i^T (\phi^{T,(l)} + \phi^{T,n})$$

Three time-level

$$R(u^{(l)}) = \sum_{T\in\Delta_i} \sum_{j\in T} m_{ij}^T \frac{3u_j^{(l)} - 4u_j^n + u_j^{n-1}}{2\Delta t} + \sum_{T\in\Delta_i} \beta_i^T \phi^{T,(l)}$$

In **non-linear** advection problem:

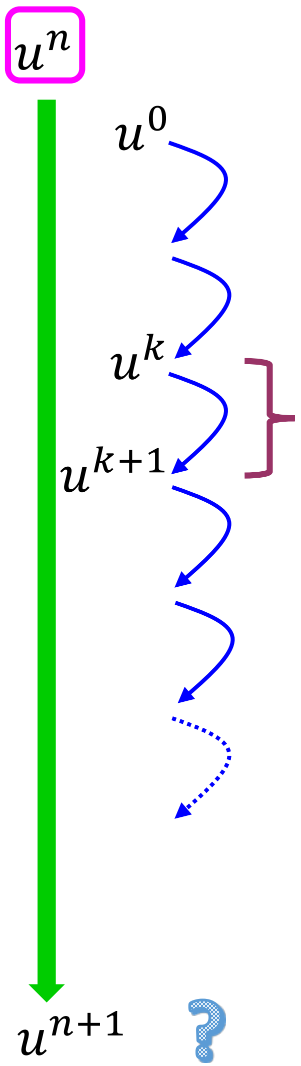
Two time-level

$$R(u^{(l)}) = \sum_{T\in\Delta_i} \sum_{j\in T} \left(m_{ij}^{T,(l)} \frac{u_j^{(l)}}{\Delta t} - m_{ij}^{T,n} \frac{u_j^n}{\Delta t} \right) + \sum_{T\in\Delta_i} \left(\frac{1}{2} \beta_i^{T,(l)} \phi^{T,(l)} + \frac{1}{2} \beta_i^{T,n} \phi^{T,n} \right)$$

Three time-level

$$R(u^{(l)}) = \sum_{T\in\Delta_i} \sum_{j\in T} \left(m_{ij}^{T,(l)} \frac{3u_j^{(l)}}{2\Delta t} - m_{ij}^{T,n} \frac{2u_j^n}{\Delta t} + m_{ij}^{T,n-1} \frac{u_j^{n-1}}{2\Delta t} \right) + \sum_{T\in\Delta_i} \beta_i^{T,(l)} \phi^{T,(l)}$$

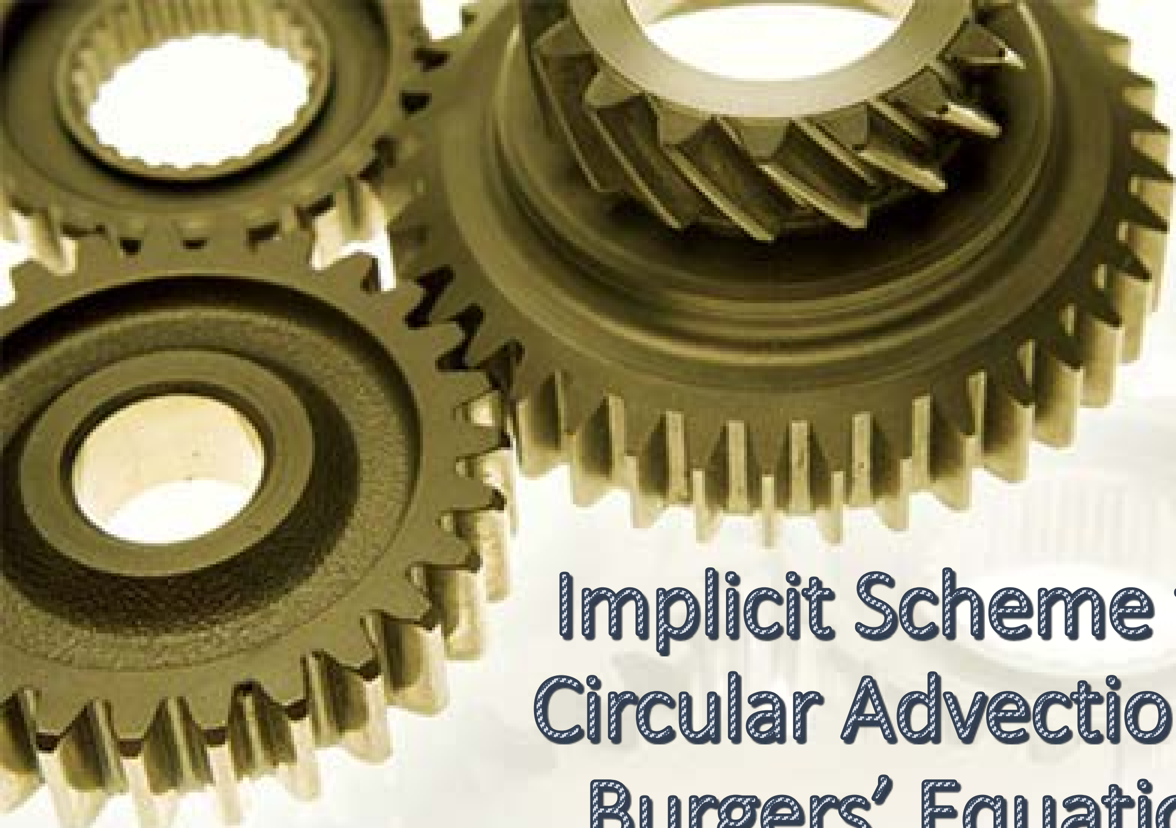
Implicit Method
(without the Five Stage Runge-Kutta Method)



$$u^{k+1} = u^k - \frac{\Delta t^*}{S_i} R(u^k)$$

Physical and Pseudo-time steps:

$$\Delta t_i^* = \Delta t_i = CFL \, \frac{2}{3} \min_{T \in \Delta_i} \frac{S_T}{\sum_{j \in T} k_j^+}$$



Implicit Scheme for Circular Advection & Burgers' Equation

- Study the validity of implicit scheme with and without the five-stage Runge-Kutta sub-iterations.
- Investigating the implicit scheme for two-time-level and also three-time-level of the time derivatives discretisation.

Circular Advection

$$\frac{\partial u}{\partial t} + (-2\pi y)\frac{\partial u}{\partial x} + (2\pi x)\frac{\partial u}{\partial y} = 0 \qquad \Omega_t = \Omega \times [0,1]$$

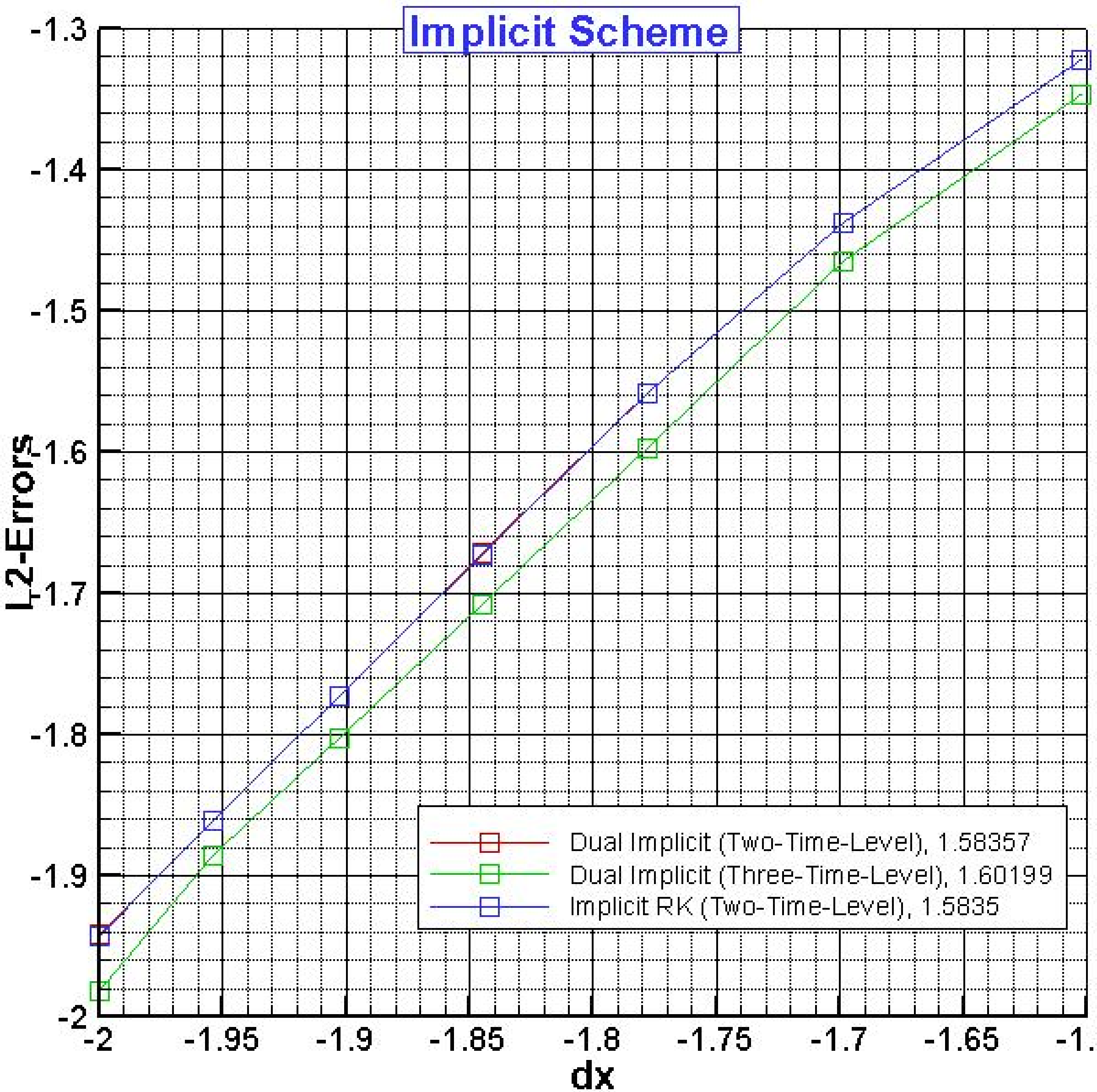
In spatial domain of $\Omega = [-1,1] \times [-1,1]$

Initial Condition:

$$u(r,0) = \begin{cases} cos^2(2\pi r), & if\ r \leq 0.25 \\ 0, & if\ r > 0.25 \end{cases} \qquad r = \sqrt{(x+0.5)^2 + y^2}$$

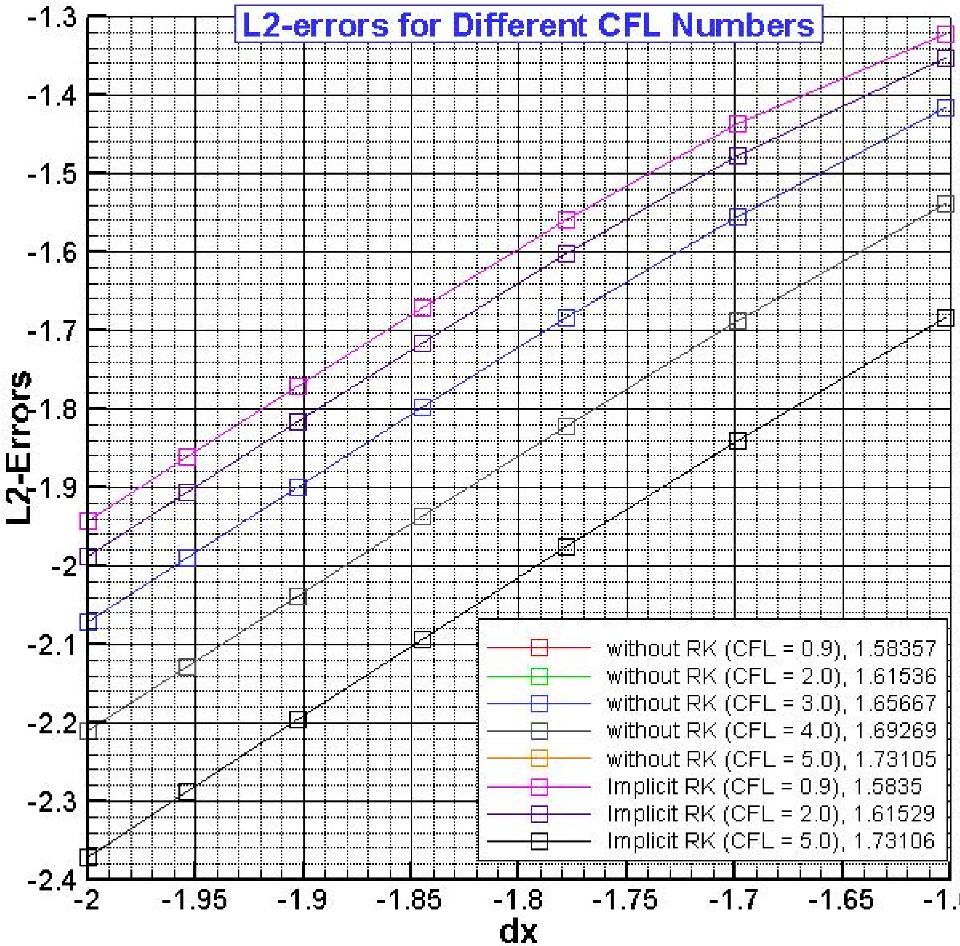
Boundary Condition:

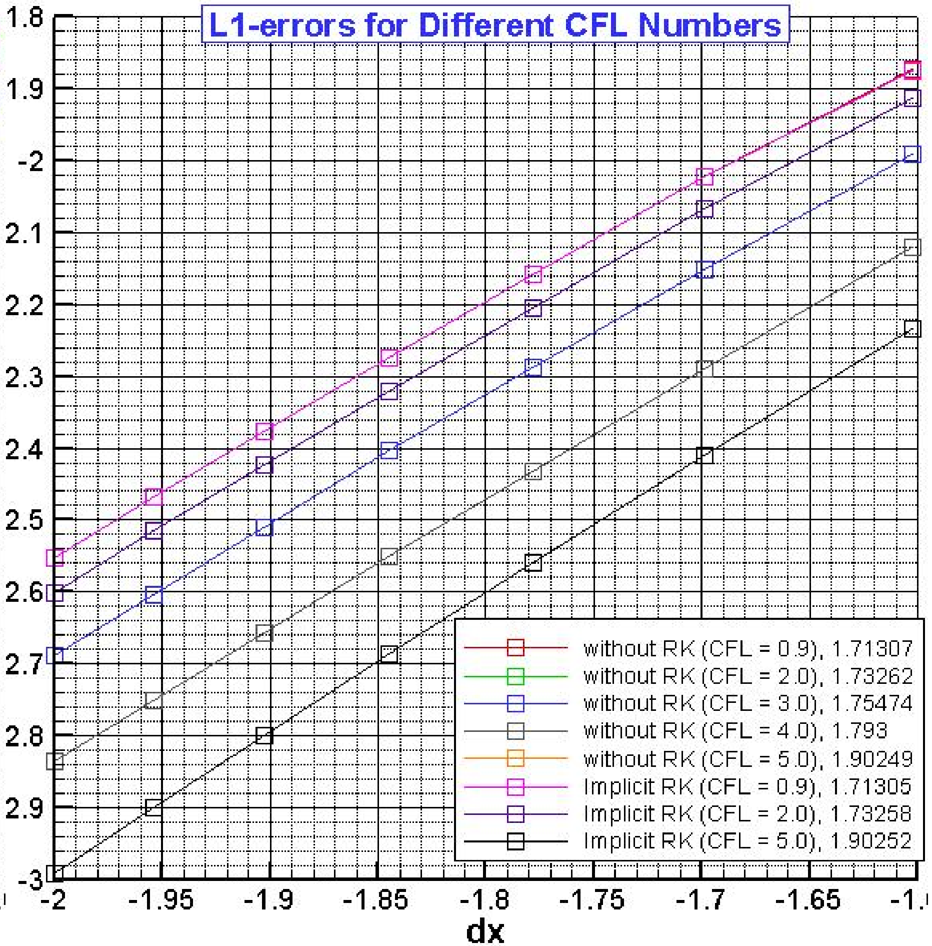
$u(r,t) = 0$ for $(x,y) \in \partial\Omega$



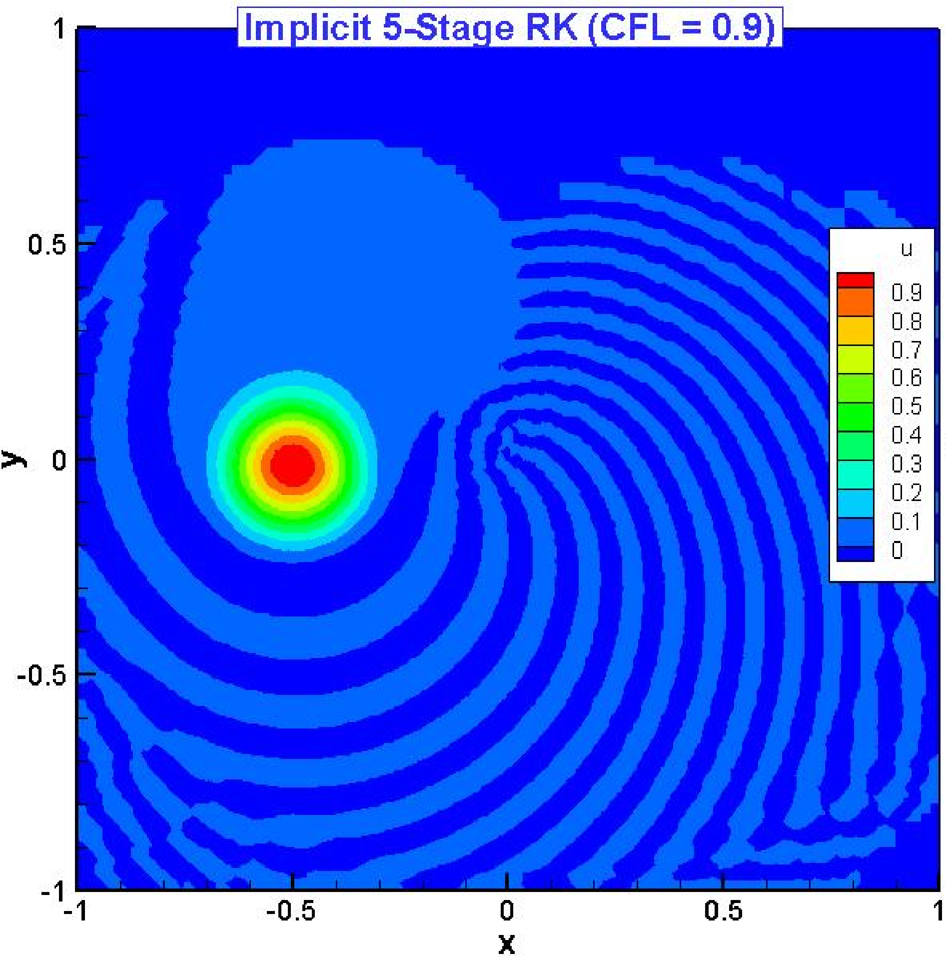
Implicit Scheme
L2-Errors
dx
-1.3
-1.4
-1.5
-1.6
-1.7
-1.8
-1.9
-2
-2
-1.95
-1.9
-1.85
-1.8
-1.75
-1.7
-1.65
Dual Implicit (Two-Time-Level), 1.58357
Dual Implicit (Three-Time-Level), 1.60199
Implicit RK (Two-Time-Level), 1.5835

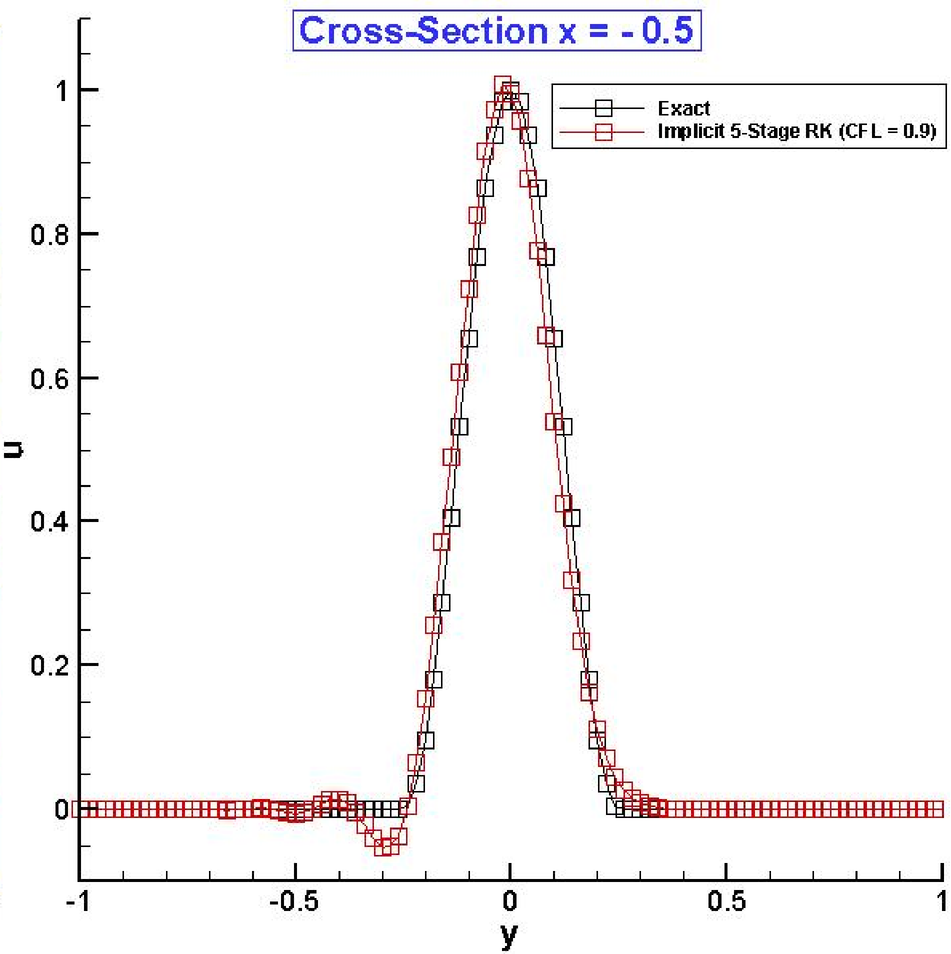
By increasing the CFL numbers from 0.9 to 5.0, there will be some effects on the L2 and L1 errors.

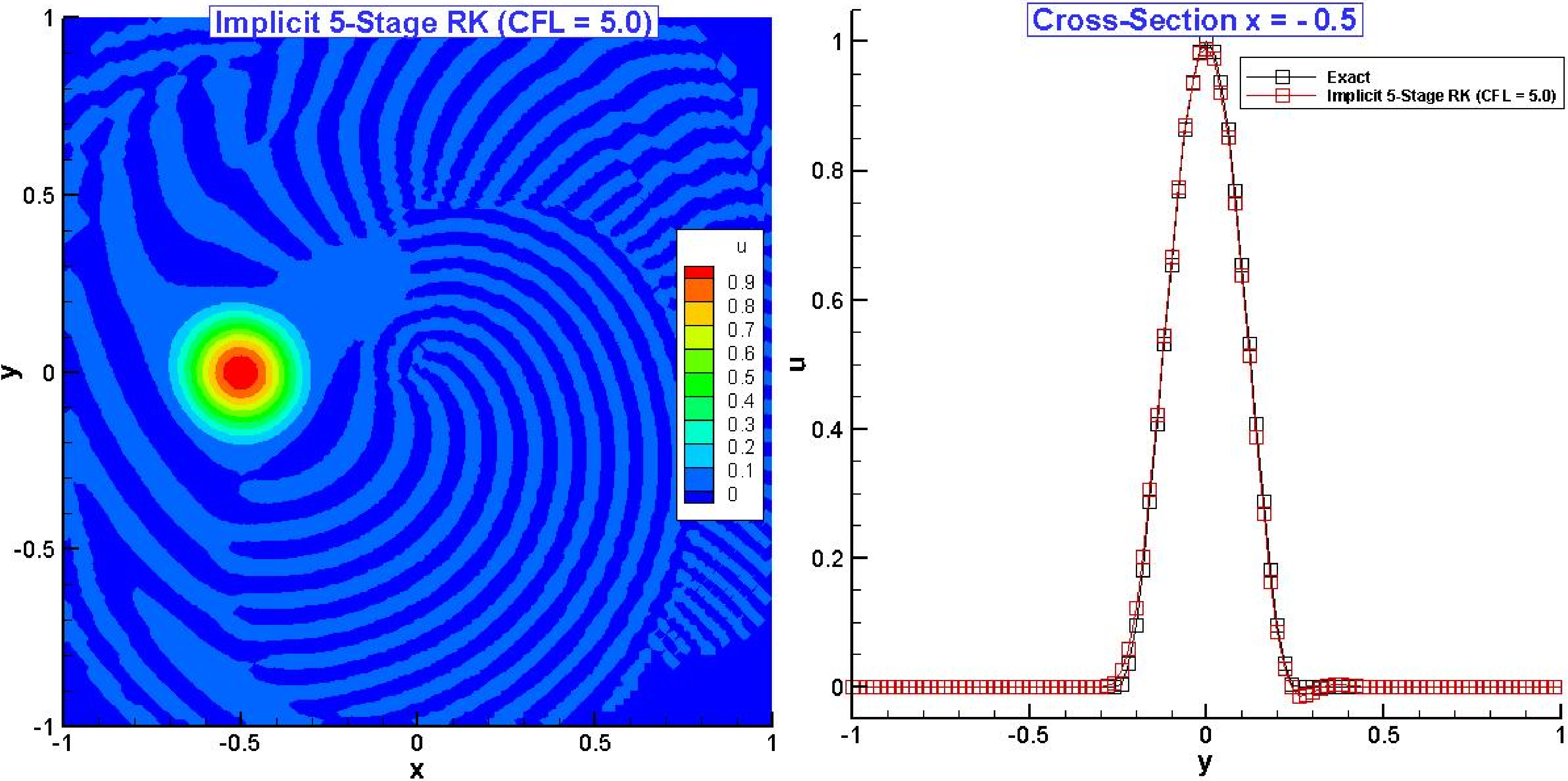




The contour plot and the cross-sectional plot for 100×100 diamond grid are plotted.







Implicit 5-Stage RK (CFL = 5.0)
u
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
0.5
0
-0.5
-1
-0.5
0
0.5
1
x
y
Cross-Section x = - 0.5
Exact
Implicit 5-Stage RK (CFL = 5.0)
1
0.8
0.6
0.4
0.2
0
u
-1
-0.5
0
0.5
1
y

Types of Solution

General case:

Weak Solution

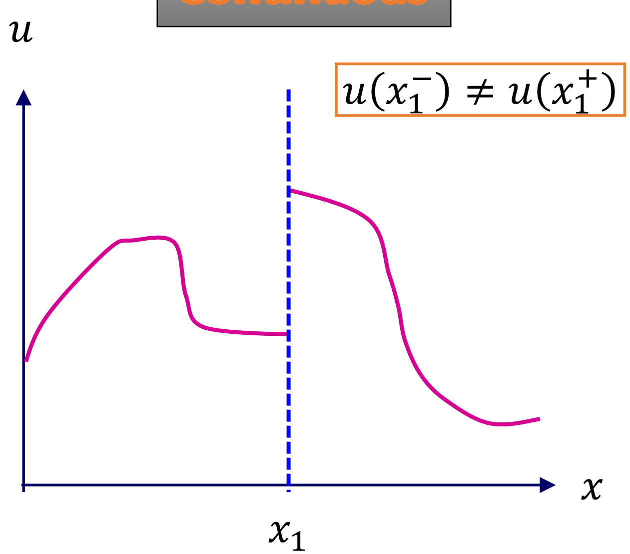
Allows discontinuous and non-smooth solutions for the PDE

Special case:

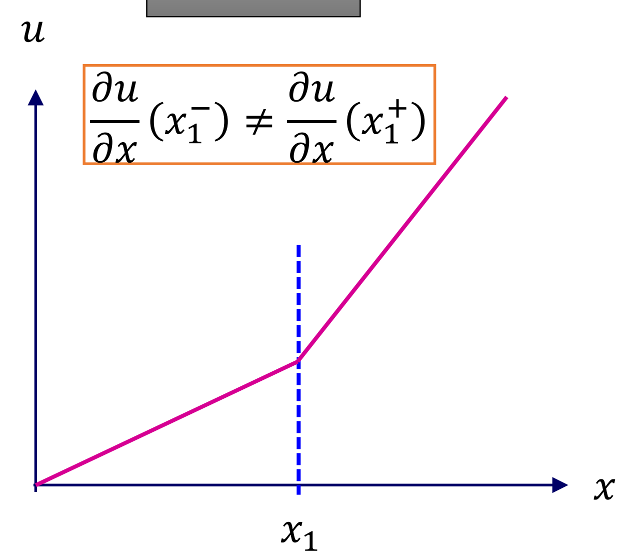
Classical Solution

The PDE must have continuous and smooth solutions

Continuous



Smooth



Weak Solution or *Generalised Solution*

- For some test case there is only *weak solution* but no *classical solution*.
- This is because the analytical (or exact) solution *might not be*
  i. continuous
  ii. smooth (differentiable)
- Riemann's problem is one typical example which has weak solution only.
- The weak solution exist as long as it fulfils the following conditions:
  i. at points of continuity

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{F} = 0$$

  ii. at points of discontinuity – *jump (shock) condition*

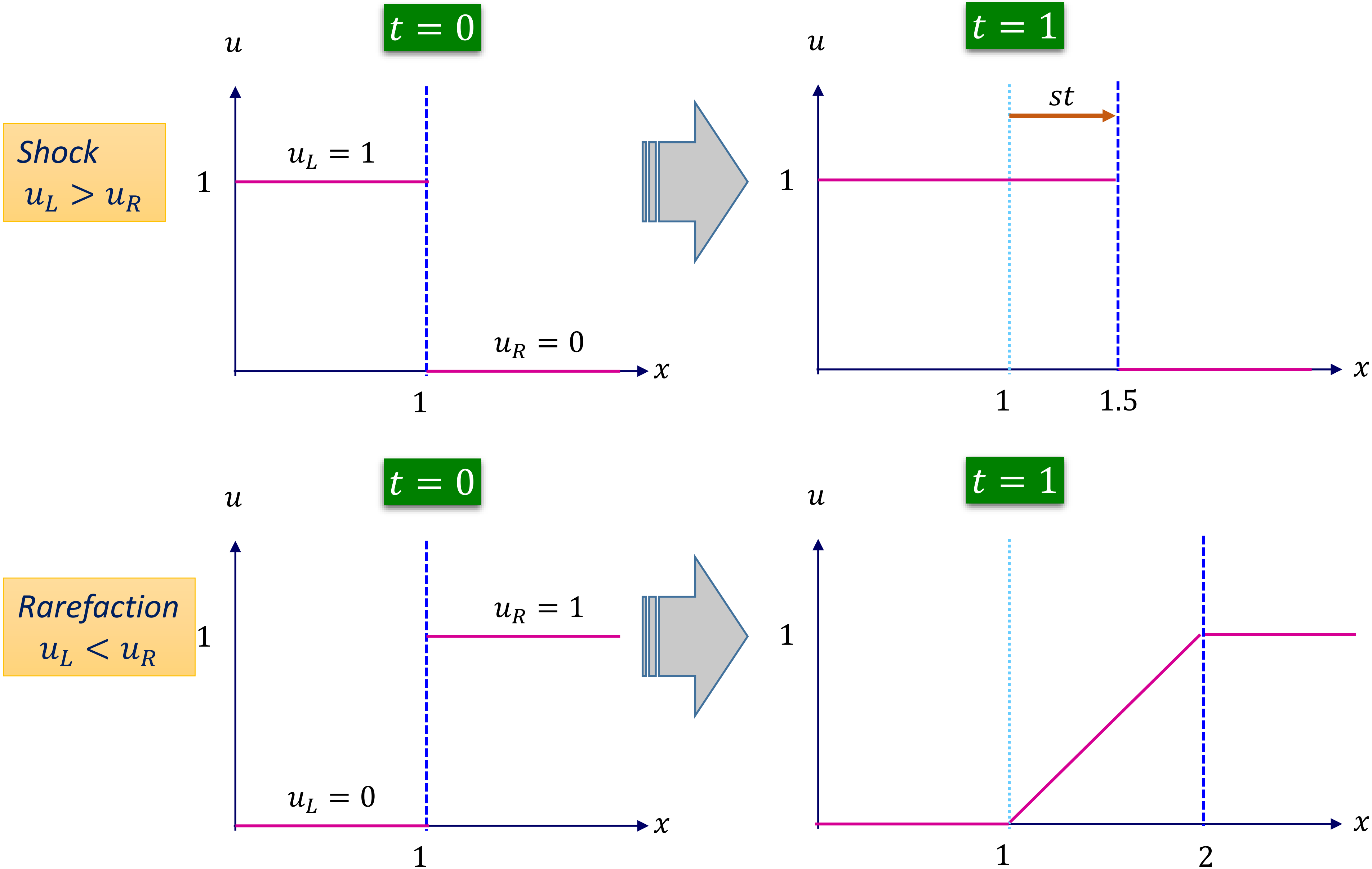
$$-s(u^{-} - u^{+}) + (F^{-} - F^{+}) = 0$$

or

$$s(u^{-} - u^{+}) = (F^{-} - F^{+})$$

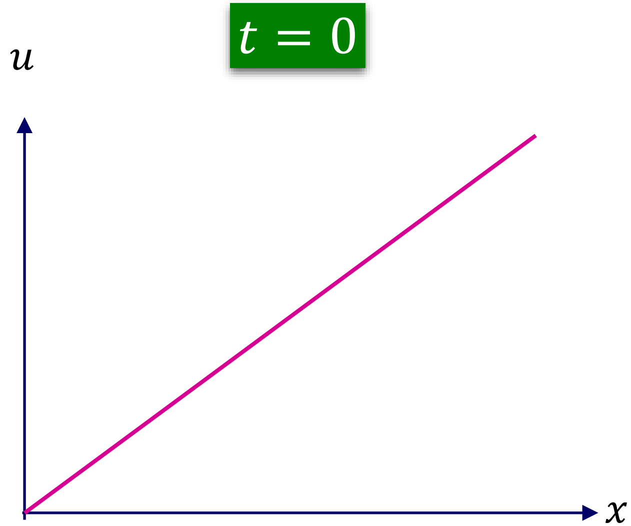
- In gas dynamics, this jump (shock) condition is known as ***Rankine-Hugoniot*** condition.

Two general test cases of Riemann's problem:

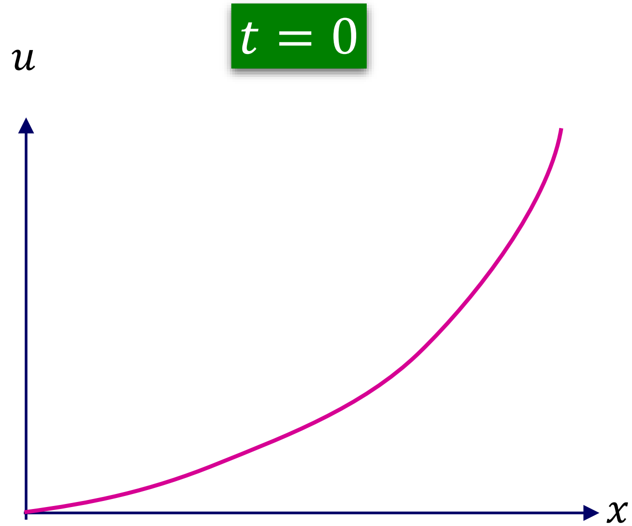


Classical Solution

- For some test case there is only *weak solution* but no *classical solution*.
- The analytical (or exact) solution *must be*
  - i. continuous
  - ii. smooth (differentiable)
- Only PDE with classical solution can be used for the second order accuracy test.
- Examples are fully expansive waves, and their initial data are:



Linear Initial Data



Quadratic Initial Data

Burgers' Equation – Linear Initial Data

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0$$

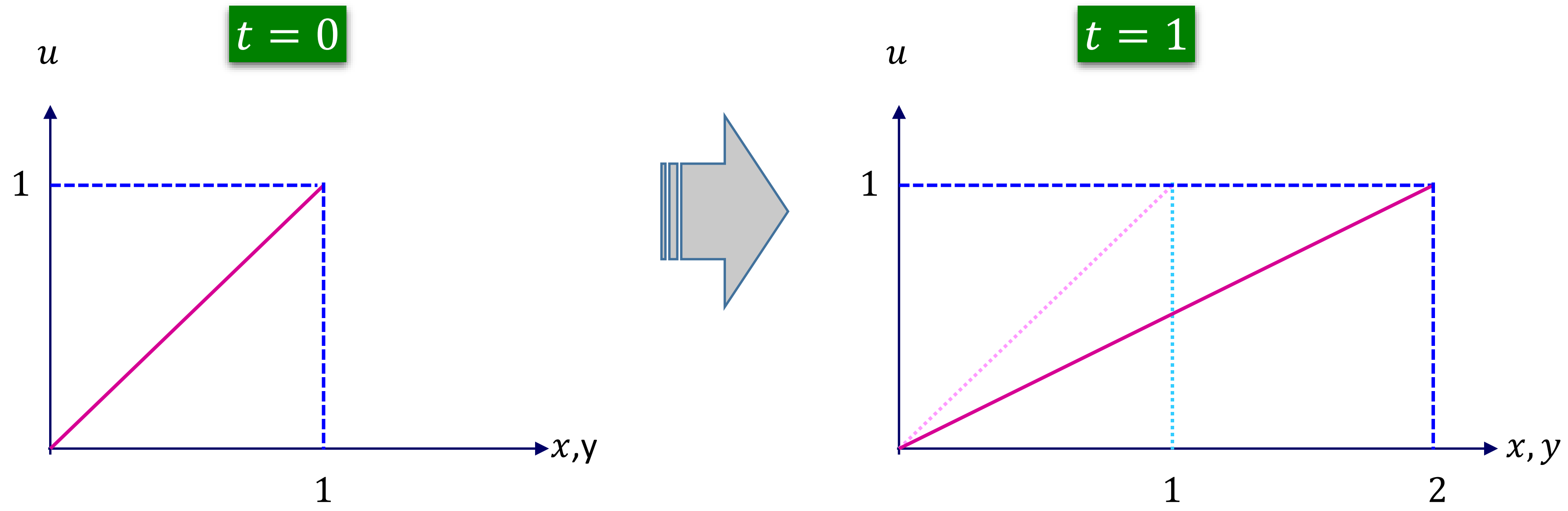
Spatial domain: $\Omega \in [0,1]^2$

Time domain: $t \in [0,1]$

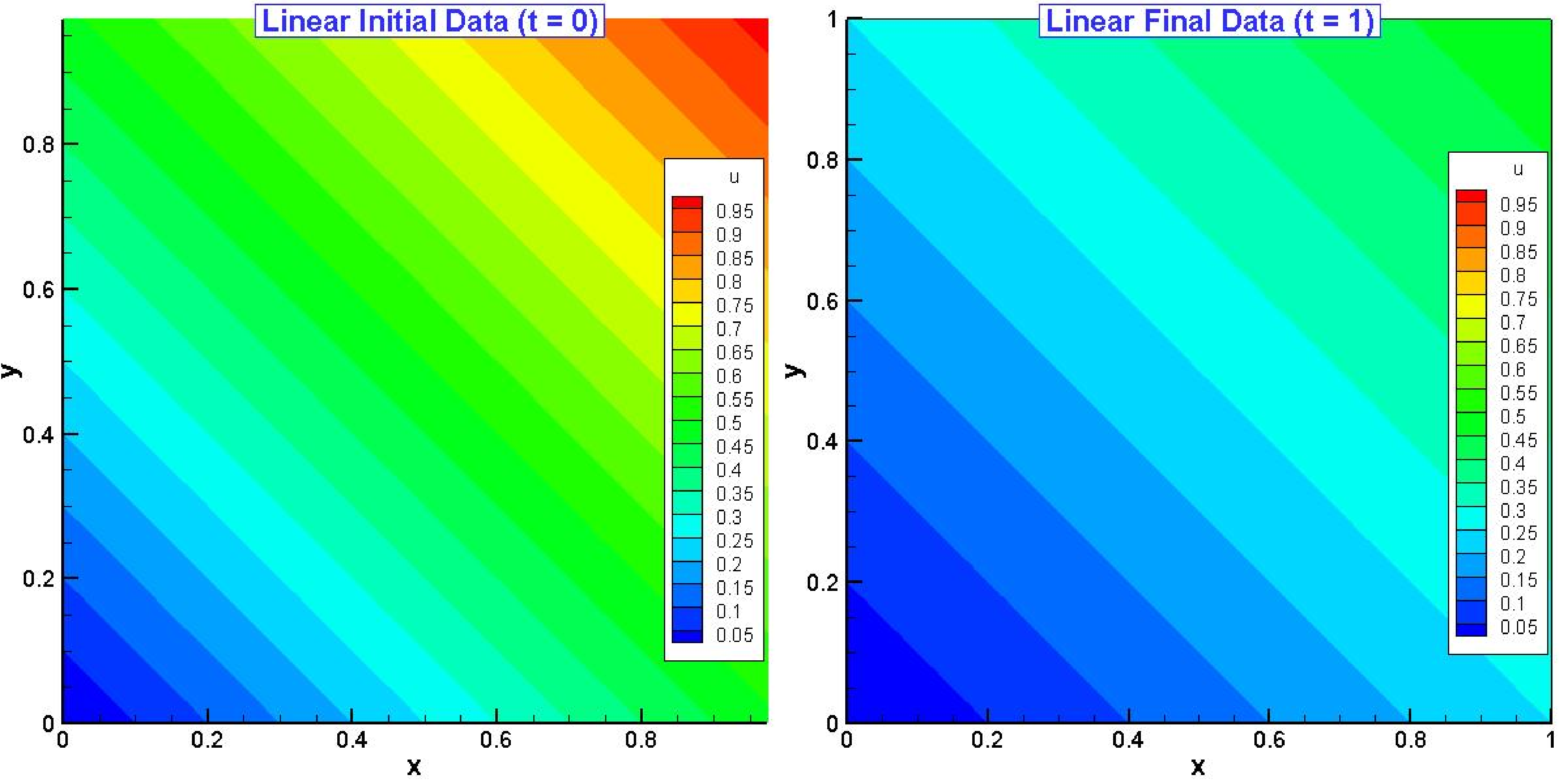
Initial data: $u(x,y,0) = \frac{x+y}{2}$

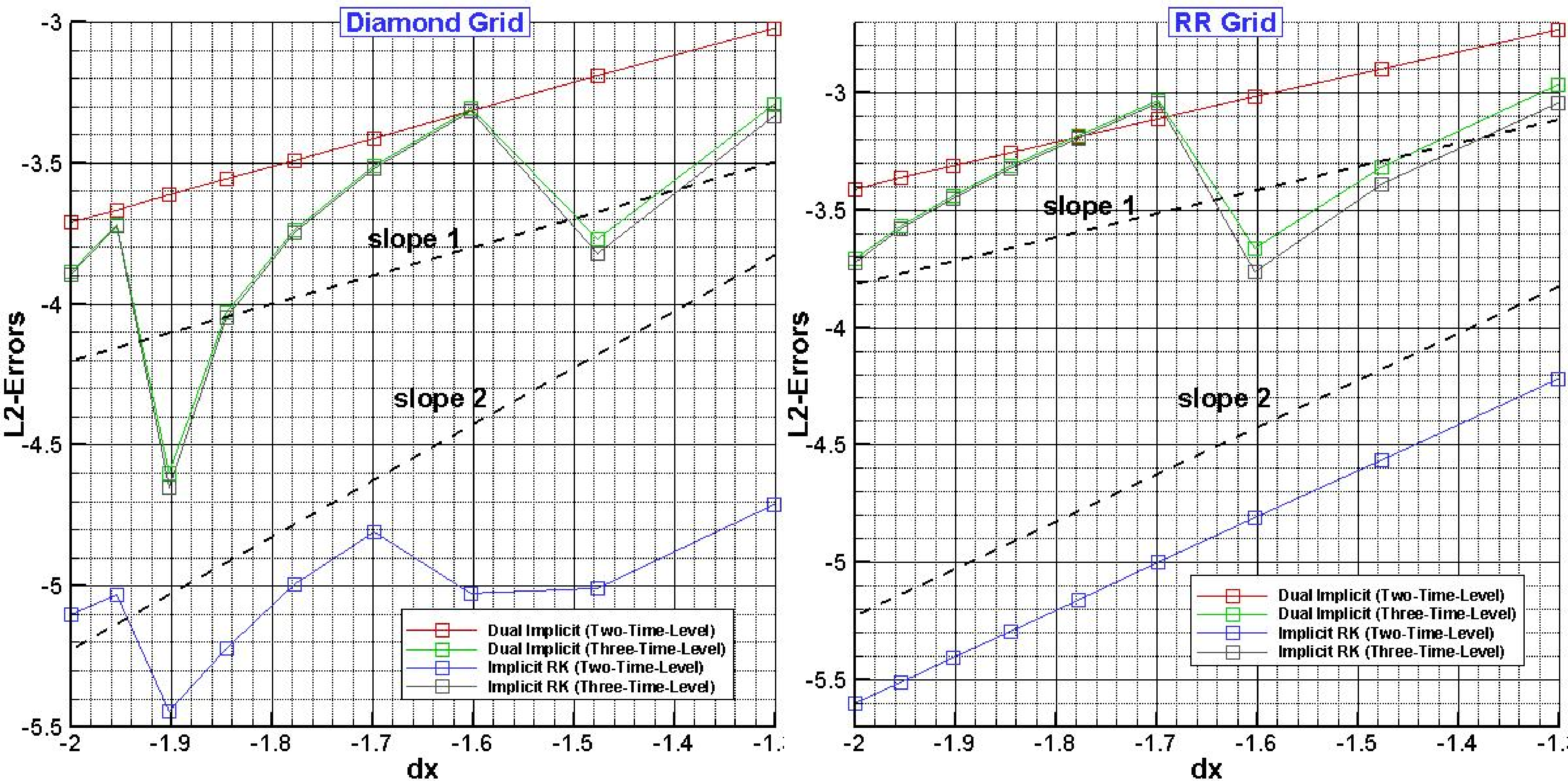
Final Solution: $u(x,y,1) = \frac{x+y}{4}$

Boundary conditions on $\partial\Omega$ for all $t \in (0,1]$: $u(x,y,t) = \frac{x+y}{2+2t}$



The initial condition and exact final solution for 100×100 diamond grid are given for linear test case.




Diamond Grid
slope 1
slope 2
L2-Errors
dx
-3
-3.5
-4
-4.5
-5
-5.5
-2
-1.9
-1.8
-1.7
-1.6
-1.5
-1.4
Dual Implicit (Two-Time-Level)
Dual Implicit (Three-Time-Level)
Implicit RK (Two-Time-Level)
Implicit RK (Three-Time-Level)
RR Grid
slope 1
slope 2
L2-Errors
dx
-3
-3.5
-4
-4.5
-5
-5.5
-2
-1.9
-1.8
-1.7
-1.6
-1.5
-1.4
Dual Implicit (Two-Time-Level)
Dual Implicit (Three-Time-Level)
Implicit RK (Two-Time-Level)
Implicit RK (Three-Time-Level)

Burgers' Equation – Quadratic Initial Data

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0$$

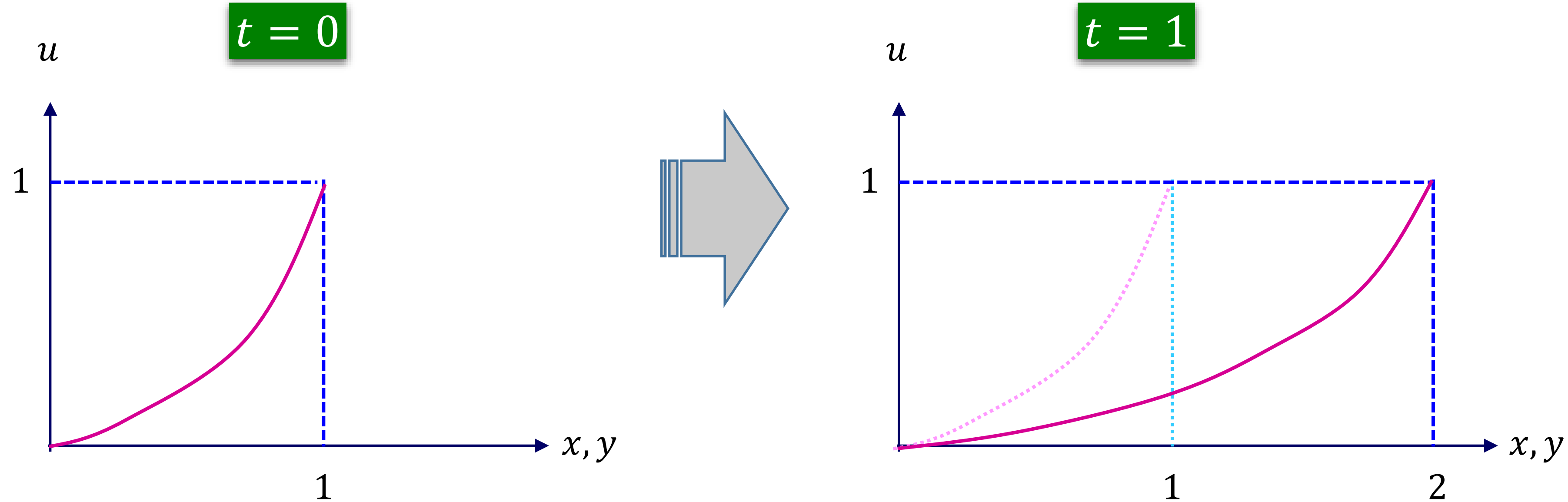
Spatial domain: $\Omega \in [0,1]^2$ **Time domain:** $t \in [0,1]$

Initial data: $u(x,y,0) = \left(\frac{x+y}{2}\right)^2$

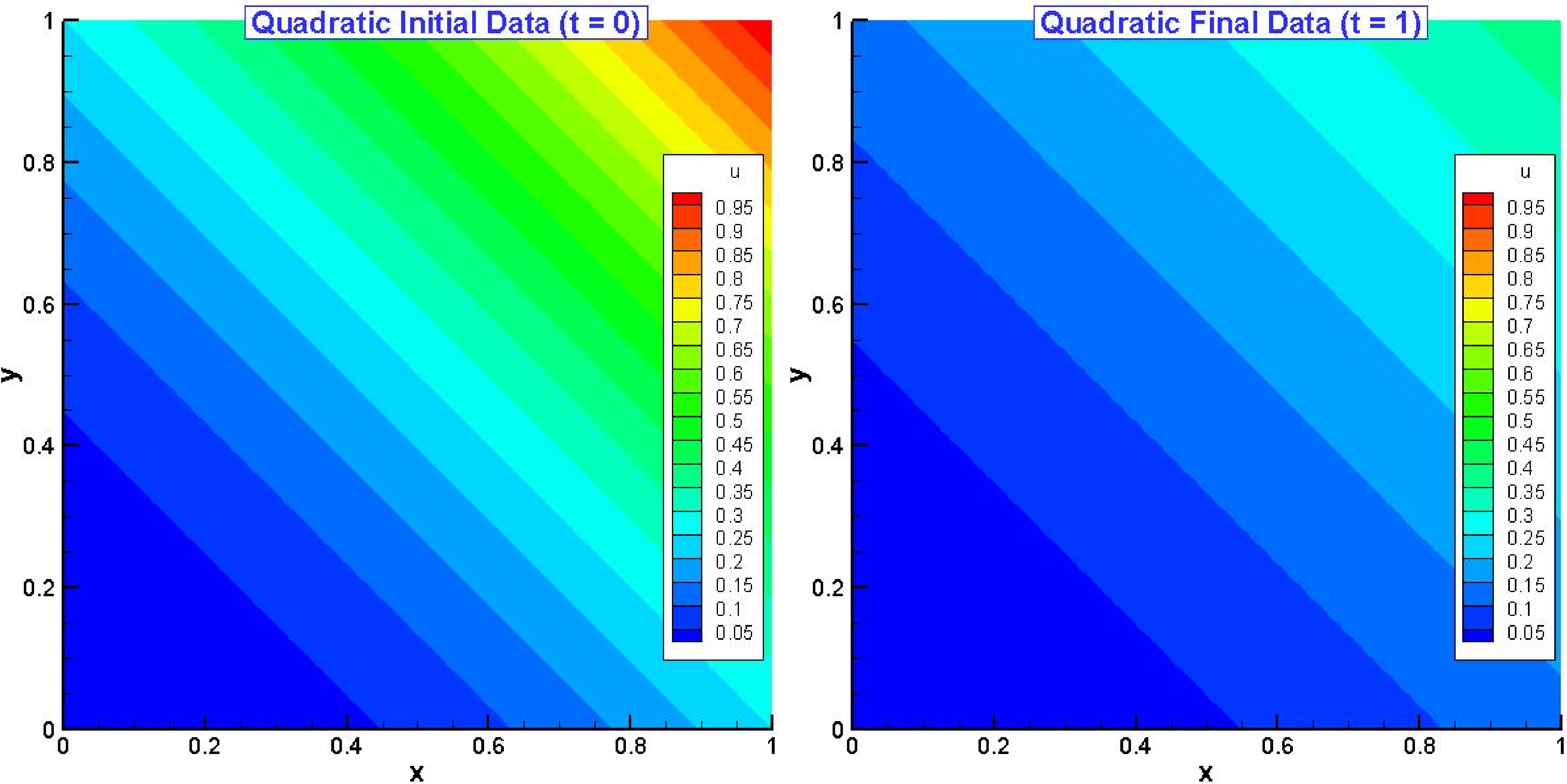
Final Solution: $u(x,y,1) = \left(-\frac{1}{2} + \frac{1}{2}\sqrt{1+2(x+y)}\right)^2$

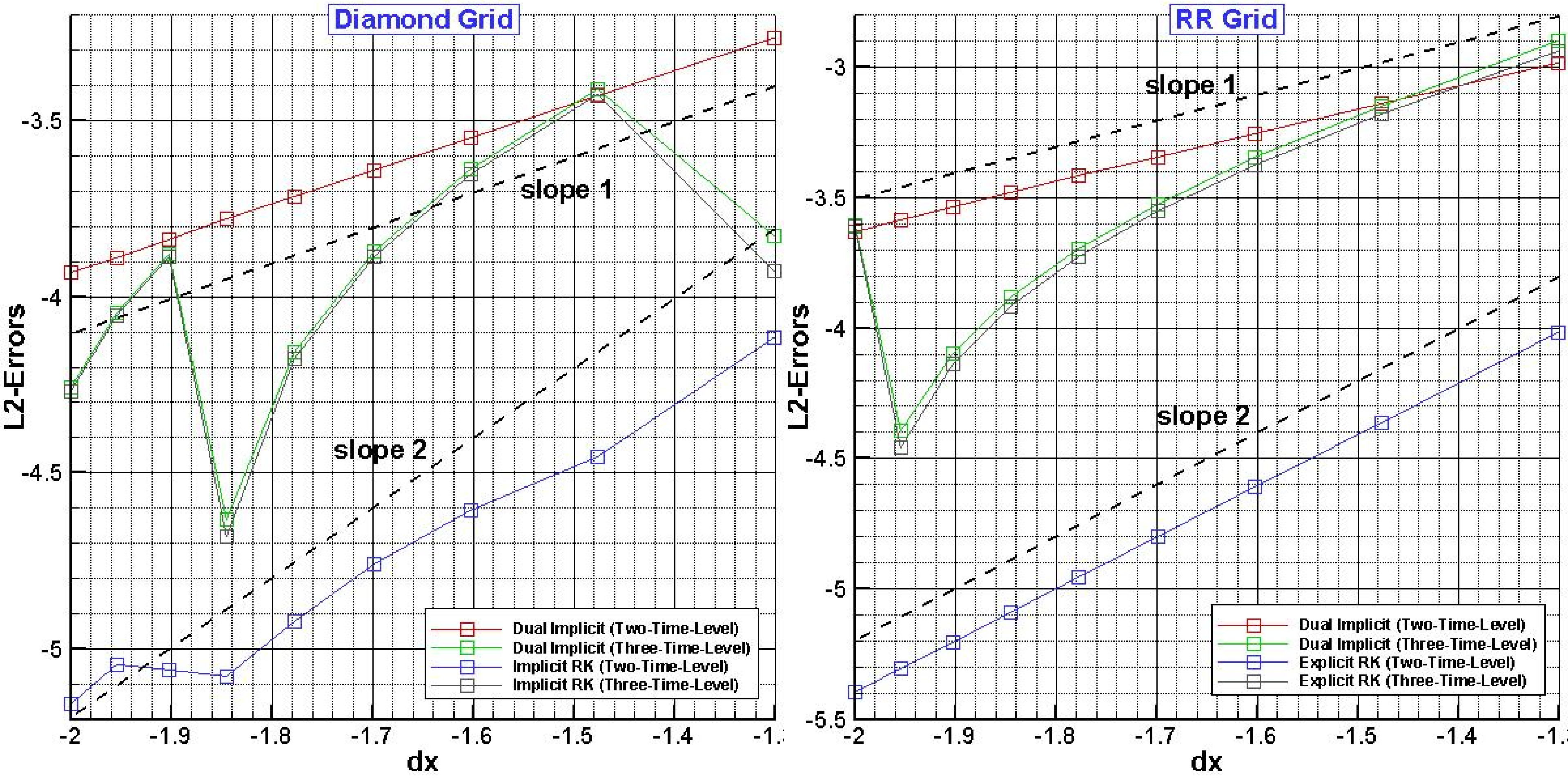
Boundary conditions on $\partial\Omega$ for all $t \in (0,1]$:

$$u(x,y,t) = \left(-\frac{1}{2t} + \frac{1}{2t}\sqrt{1+2t(x+y)}\right)^2$$



The initial condition and exact final solution for 100×100 diamond grid are given for quadratic test case.




Diamond Grid
slope 1
slope 2
L2-Errors
dx
Dual Implicit (Two-Time-Level)
Dual Implicit (Three-Time-Level)
Implicit RK (Two-Time-Level)
Implicit RK (Three-Time-Level)
RR Grid
slope 1
slope 2
L2-Errors
dx
Dual Implicit (Two-Time-Level)
Dual Implicit (Three-Time-Level)
Explicit RK (Two-Time-Level)
Explicit RK (Three-Time-Level)

Explicit Runge-Kutta for Time-Dependent Problem

- This is the work proposed by Ricchiuto & Abgrall (2010).
- The explicit scheme can be implemented in two ways, either two-stage Runge-Kutta or three-stage Runge-Kutta intermediate steps. We will emphasise only on he former.

Explicit Two-Stage Runge-Kutta Method

Selectively-Lumped (RK_SL)

Stage 1:

$$S_i \frac{u_i^1 - u_i^n}{\Delta t} = -\sum_{T \in \Delta_i} \beta_i^T \phi^T (u^n)$$

Stage 2:

$$S_i \frac{u_i^{n+1} - u_i^n}{\Delta t} = -\sum_{T \in \Delta_i} \left\{ \sum_{j \in T} m_{ij}^T \frac{u_i^1 - u_i^n}{\Delta t} - \sum_{j \in T} m_{ij}^G \frac{u_i^1 - u_i^n}{\Delta t} + \frac{1}{2} \beta_i^T \left(\phi^T (u^n) + \phi^T (u^1) \right) \right\}$$

Globally-Lumped (RK_GL)

Stage 1:

$$S_i \frac{u_i^1 - u_i^n}{\Delta t} = -\sum_{T \in \Delta_i} \beta_i^T \phi^T (u^n)$$

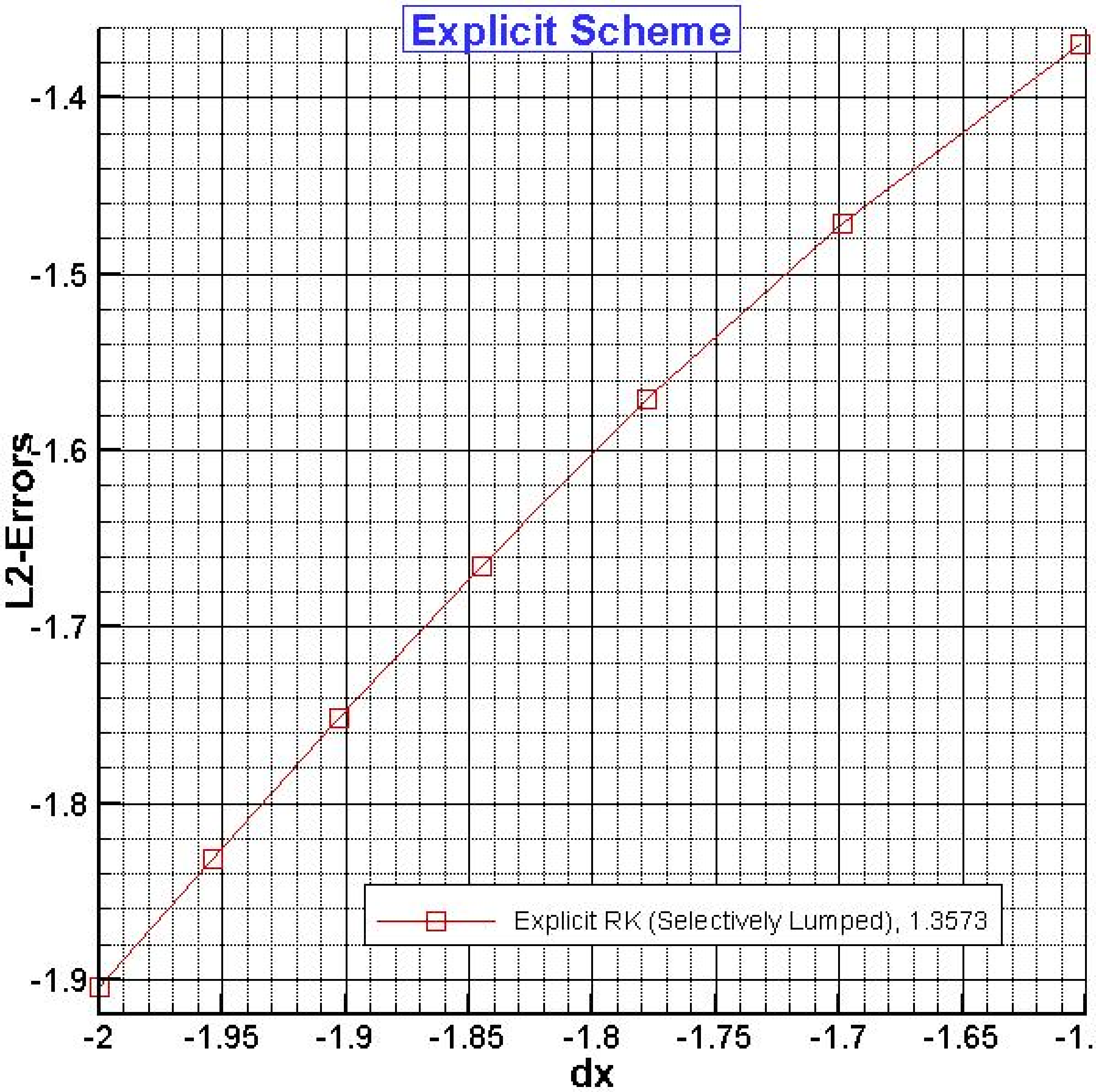
Stage 2:

$$S_i \frac{u_i^{n+1} - u_i^1}{\Delta t} = -\sum_{T \in \Delta_i} \left\{ \sum_{j \in T} m_{ij}^T \frac{u_i^1 - u_i^n}{\Delta t} + \frac{1}{2} \beta_i^T \left(\phi^T (u^n) + \phi^T (u^1) \right) \right\}$$

where $m_{ij}^T = \frac{S_T}{3} \beta_i \left(\delta_{ij} + 1 - \beta_j \right)$ $\quad m_{ij}^G = \frac{S_T}{12} \left(\delta_{ij} + 1 \right)$

consistent upwind mass matrix

Circular Advection



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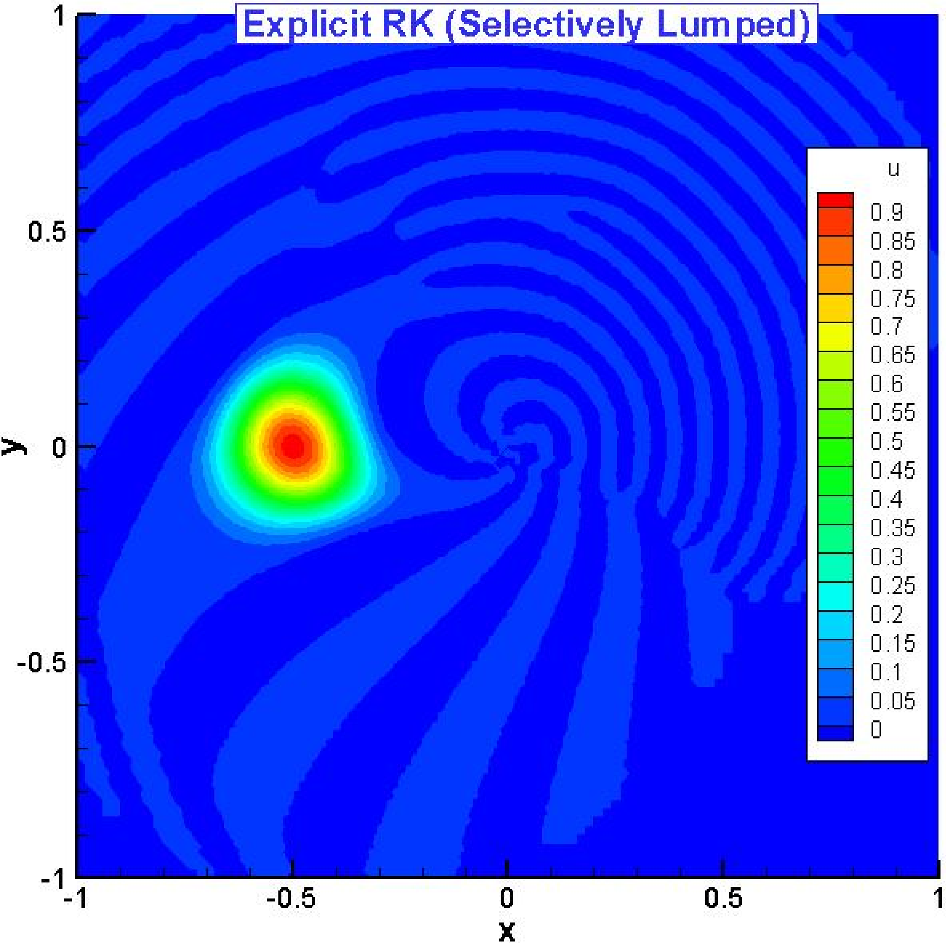
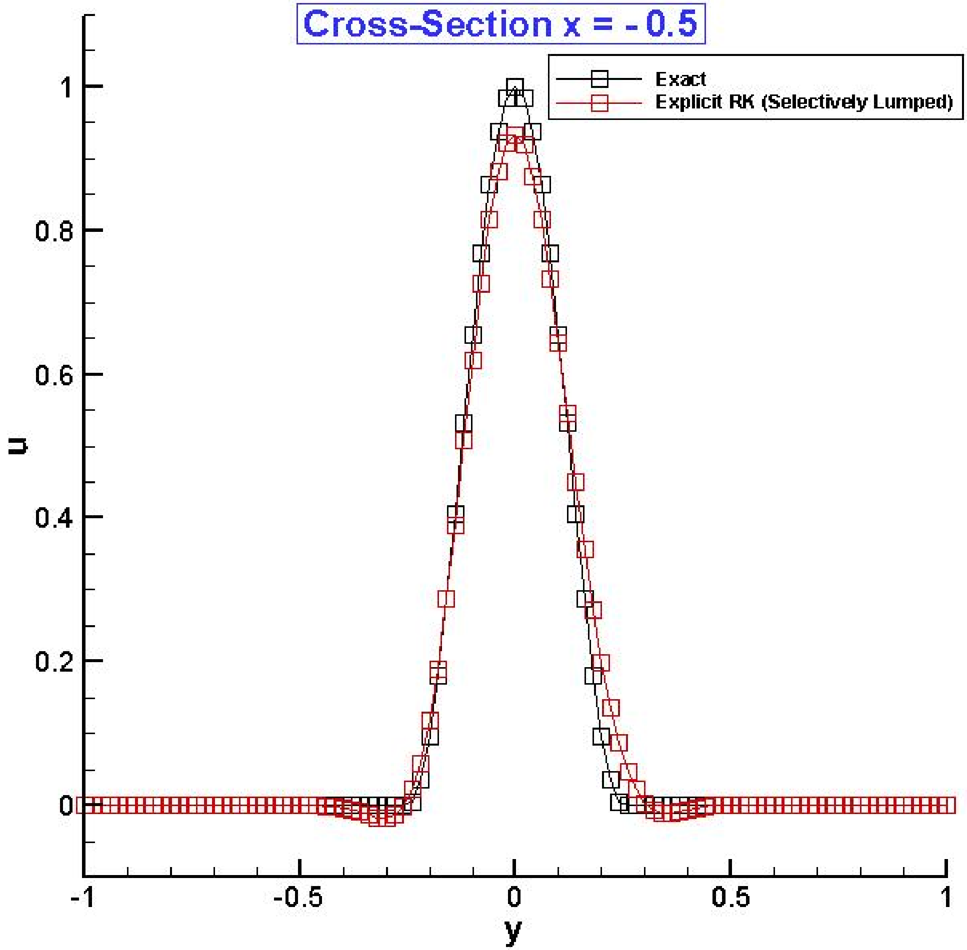
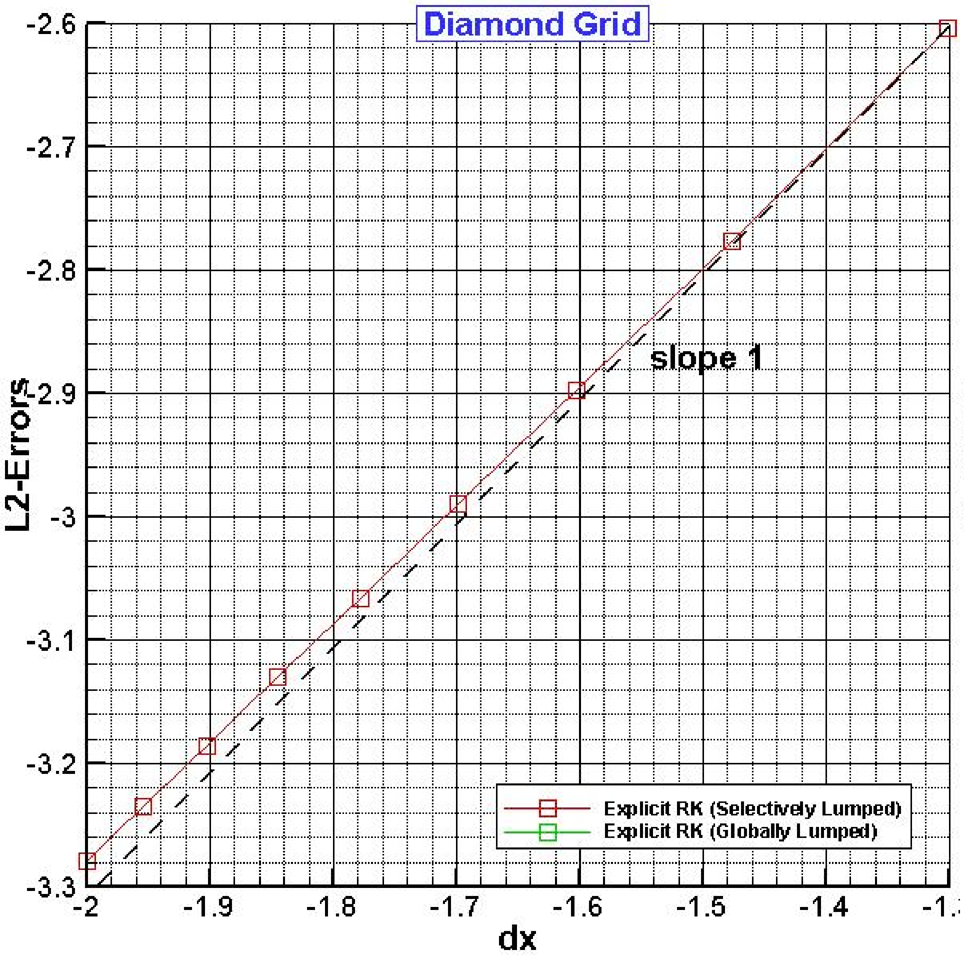
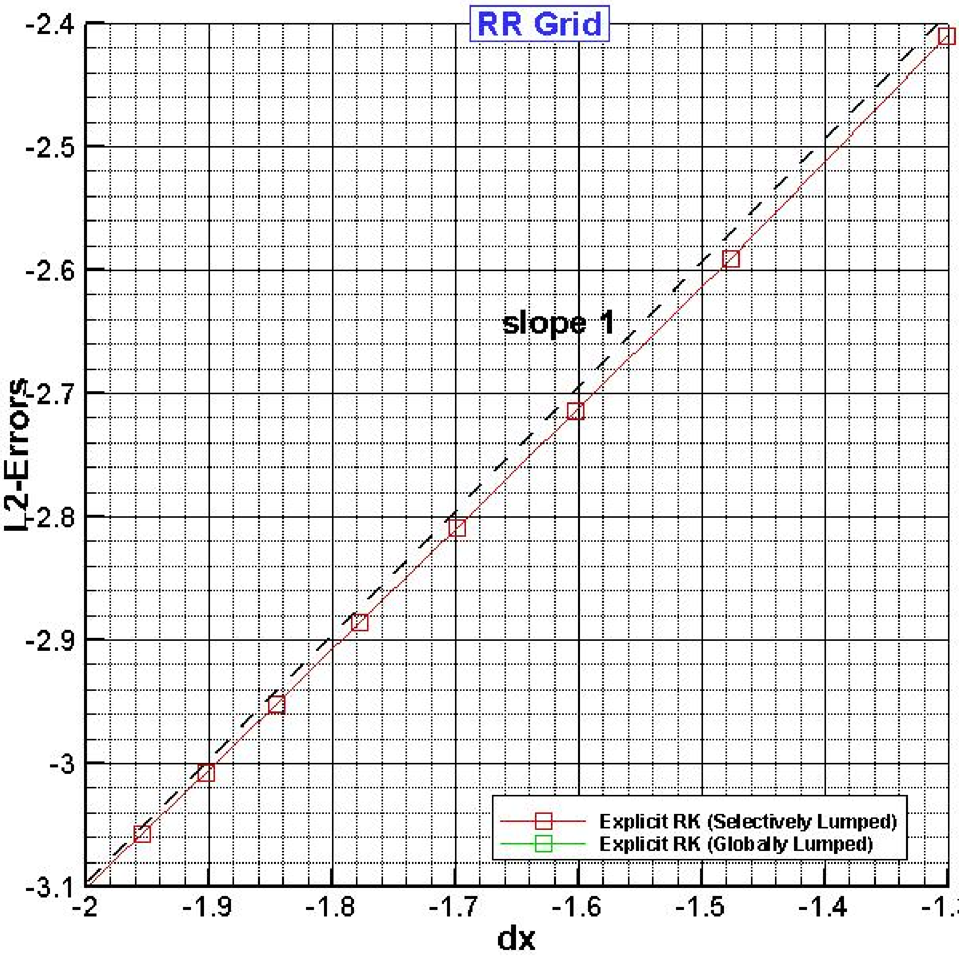




Burgers' Equation – Linear Initial Data





Burgers' Equation – Quadratic Initial Data