2D Unsteady Maxwell's Equations

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2D Maxwell’s Equations & Inflow Matrices

The Complete 3D Maxwell's Equations

$$\vec{B} = \mu \vec{H}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \Longrightarrow \qquad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \mu \vec{J} + \varepsilon\mu \frac{\partial \vec{H}}{\partial t} \qquad \Longrightarrow \qquad \nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{H}}{\partial t}$$

Assuming there is no source current, $\vec{J} = 0$

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right) \qquad \frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) \qquad \frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) \qquad \frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

Recasting the 2D Maxwell's equations in a form similar to conservation law gives

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\frac{\partial H_z}{\partial y}$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\varepsilon}\frac{\partial H_z}{\partial x}$$

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu}\frac{\partial E_y}{\partial x} + \frac{1}{\mu}\frac{\partial E_x}{\partial y}$$

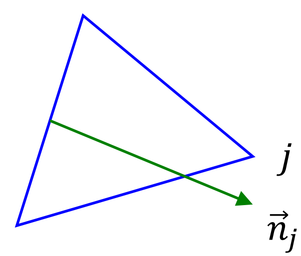
$$\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{\mathbb{F}} = 0$$

$$\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \mathbb{A}\vec{U} = 0 \qquad \mathbb{A} = \left(A_x, A_y\right)$$

$$\frac{\partial \vec{U}}{\partial t} + \mathbb{A} \cdot \nabla\vec{U} = 0$$

The conserved variables $\vec{U} = \left(E_x, E_y, H_z\right)$

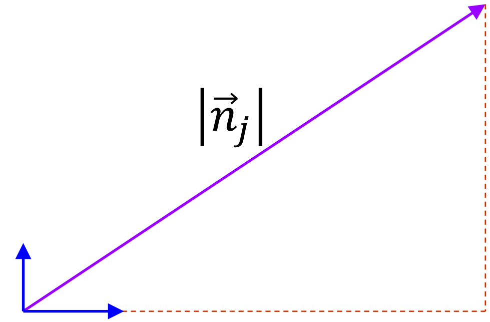
$$A_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\varepsilon} \\ 0 & -\frac{1}{\mu} & 0 \end{pmatrix} \qquad A_y = \begin{pmatrix} 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & 0 \\ \frac{1}{\mu} & 0 & 0 \end{pmatrix}$$



Inflow matrix $\mathbb{K}_j = \frac{1}{2}\left(A_x n_{jx} + A_y n_{jy}\right)$

$$\vec{n}_j = n_{jx}\hat{x} + n_{jy}\hat{y}$$

$$\vec{n}_j = (\eta_{jx}|\vec{n}_j|)\hat{x} + (\eta_{jy}|\vec{n}_j|)\hat{y}$$



$$\eta_{jy}\hat{y} = \frac{n_{jy}}{|\vec{n}_j|}\hat{y}$$

$$\eta_{jx}\hat{x} = \frac{n_{jx}}{|\vec{n}_j|}\hat{x}$$

$\eta_{jx}\hat{x}$ and $\eta_{jy}\hat{y}$ being the unit vectors.

$|\vec{n}_j|$ being the magnitude of the inward scaled normal.

The reason of simplifying the notation of the scaled inward normal is such that we can use it to simplify the flux-splitting of inflow matrix due to

$$\eta_{jx}{}^2 + \eta_{jy}{}^2 = 1$$

The inflow matrix can be diagonalised such as $\mathbb{K}_j = \frac{|\vec{n}_j|}{2}\mathbb{R}\Lambda\mathbb{R}^{-1}$

Set of eigenvectors : right-eigenvector

$$\mathbb{R} = \begin{pmatrix} -\eta_{jy}\sqrt{\frac{\mu}{\varepsilon}} & \eta_{jx} & \eta_{jy}\sqrt{\frac{\mu}{\varepsilon}} \\ \eta_{jx}\sqrt{\frac{\mu}{\varepsilon}} & \eta_{jy} & -\eta_{jx}\sqrt{\frac{\mu}{\varepsilon}} \\ 1 & 0 & 1 \end{pmatrix}$$

Eigenvalues

$$\Lambda = \begin{pmatrix} -c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}$$

Speed of light

$$c = (\mu\varepsilon)^{-\frac{1}{2}}$$

left-eigenvectors

Intrinsic impedance

$$\mathbb{R}^{-1} = \frac{1}{2Z}\begin{pmatrix} -\eta_{jy} & \eta_{jx} & Z \\ 2\eta_{jx}Z & 2\eta_{jy}Z & 0 \\ \eta_{jy} & -\eta_{jx} & Z \end{pmatrix}$$

$$Z = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\mathbb{K}_j^{+} = \frac{|\vec{n}_j|}{2}\mathbb{R}\Lambda^{+}\mathbb{R}^{-1} = \frac{|\vec{n}_j|}{4Z}\begin{pmatrix} c{\eta_{jy}}^2 Z & -c\eta_{jx}\eta_{jy}Z & c\eta_{jy}Z^2 \\ -c\eta_{jx}\eta_{jy}Z & c{\eta_{jx}}^2 Z & -c\eta_{jx}Z^2 \\ c\eta_{jy} & -c\eta_{jx} & cZ \end{pmatrix}$$

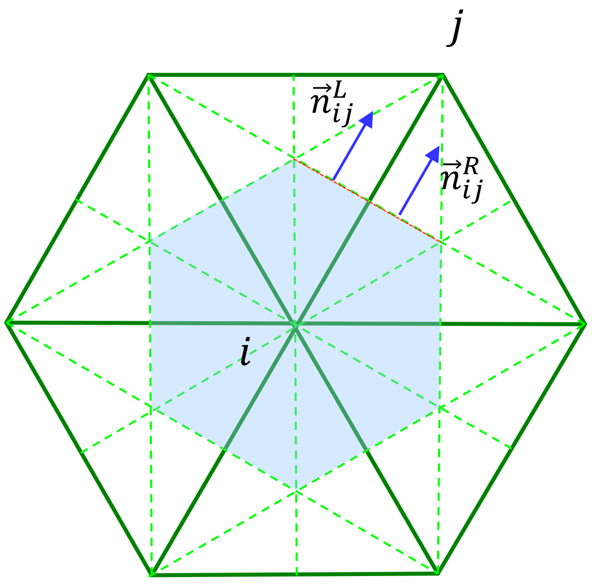
$$\mathbb{K}_j^{-} = \frac{|\vec{n}_j|}{2}\mathbb{R}\Lambda^{-}\mathbb{R}^{-1} = \frac{|\vec{n}_j|}{4Z}\begin{pmatrix} -c{\eta_{jy}}^2 Z & c\eta_{jx}\eta_{jy}Z & c\eta_{jy}Z^2 \\ c\eta_{jx}\eta_{jy}Z & -c{\eta_{jx}}^2 Z & -c\eta_{jx}Z^2 \\ c\eta_{jy} & -c\eta_{jx} & -cZ \end{pmatrix}$$

1st Order Upwind FV & RD Scheme

1st Order Upwind FV

inflow matrix

$$\mathbb{K}_{ij}^{L} = \frac{1}{2}\frac{\partial \vec{\mathbb{F}}}{\partial \vec{U}} \cdot \vec{n}_{ij}^{L} = \frac{\left|\vec{n}_{ij}^{L}\right|}{2}\mathbb{R}\Lambda\mathbb{R}^{-1}$$



Absolute inflow matrix

$$\left|\mathbb{K}_{ij}^{L}\right| = \frac{\left|\vec{n}_{ij}^{L}\right|}{2}\mathbb{R}|\Lambda|\mathbb{R}^{-1} = \frac{\left|\vec{n}_{ij}^{L}\right|}{2}\begin{pmatrix} c{\eta_{ij,y}^{L}}^{2} & -c\eta_{ij,x}^{L}\eta_{ij,y}^{L} & 0 \\ -c\eta_{ij,x}^{L}\eta_{ij,y}^{L} & c{\eta_{ij,x}^{L}}^{2} & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$S_i\frac{\vec{U}_i^{n+1} - \vec{U}_i^{n}}{\Delta t} = -\sum_{j\in\{k\}_i}\left[\vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^{L}\right) + \vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^{R}\right)\right]$$

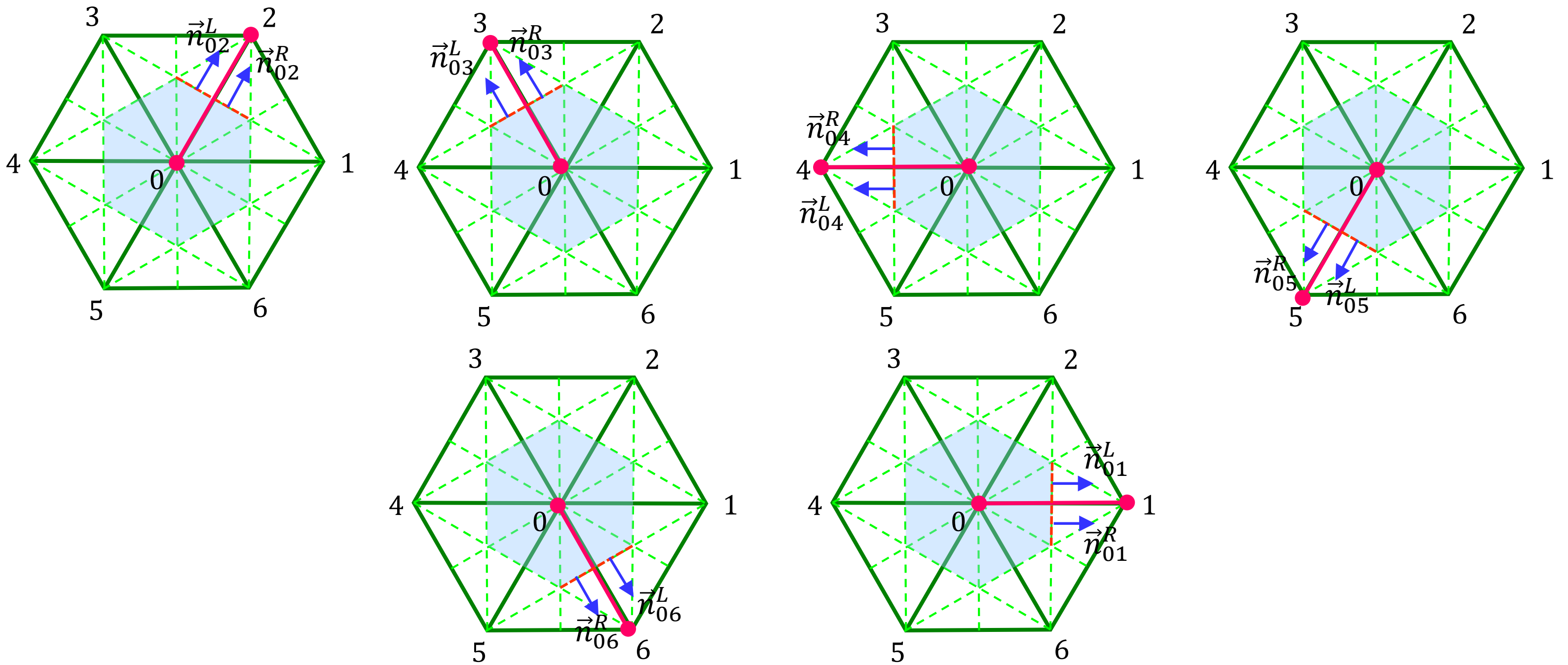
$$\vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^{L}\right) = \frac{\vec{\mathbb{F}}\left(\vec{U}_i\right)\cdot\vec{n}_{ij}^{L} + \vec{\mathbb{F}}\left(\vec{U}_j\right)\cdot\vec{n}_{ij}^{L}}{2} - \left|\mathbb{K}_{ij}^{L}\right|\left(\vec{U}_j - \vec{U}_i\right)$$

$$\vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^{R}\right) = \frac{\vec{\mathbb{F}}\left(\vec{U}_i\right)\cdot\vec{n}_{ij}^{R} + \vec{\mathbb{F}}\left(\vec{U}_j\right)\cdot\vec{n}_{ij}^{R}}{2} - \left|\mathbb{K}_{ij}^{R}\right|\left(\vec{U}_j - \vec{U}_i\right)$$

$$S_i\frac{\vec{U}_i^{n+1} - \vec{U}_i^{n}}{\Delta t} = -\sum_{j\in\{k\}_i}\left[\mathbb{K}_{ij}^{+}\vec{U}_i^{n} + \mathbb{K}_{ij}^{-}\vec{U}_j^{n}\right]$$

by the use of the identity

$$\mathbb{K}_{ij}^{\pm} = \left(\mathbb{K}_{ij}^{L} \pm \left|\mathbb{K}_{ij}^{L}\right|\right) + \left(\mathbb{K}_{ij}^{R} \pm \left|\mathbb{K}_{ij}^{R}\right|\right)$$



3
2
$\vec{n}_{02}^{L}$
$\vec{n}_{02}^{R}$
4
0
1
5
6
3
2
$\vec{n}_{03}^{L}$
$\vec{n}_{03}^{R}$
4
0
1
5
6
3
2
$\vec{n}_{04}^{R}$
4
0
1
$\vec{n}_{04}^{L}$
5
6
3
2
4
0
1
$\vec{n}_{05}^{R}$
$\vec{n}_{05}^{L}$
5
6
3
2
4
0
1
$\vec{n}_{06}^{L}$
$\vec{n}_{06}^{R}$
5
6
3
2
$\vec{n}_{01}^{L}$
4
0
1
$\vec{n}_{01}^{R}$
5
6

RD Scheme

N-Scheme :

$$S_i \frac{\vec{U}_i^{n+1} - \vec{U}_i^n}{\Delta t} = - \sum_{T \in \cup \Delta_i} \Phi_i^T(\vec{U}^n)$$

$$\mathbb{N} = \left(\sum_{j \in T} \mathbb{K}_j^+ \right)^{-1}$$

$$\Phi_i^T(\vec{U}^n) = \mathbb{K}_i^+ \left(\vec{U}_i^n + \mathbb{N} \sum_{j \in T} \mathbb{K}_j^- \vec{U}_j^n \right)$$

$$\underbrace{+\mathbb{N} \sum_{j \in T} \mathbb{K}_j^- \vec{U}_j^n}_{-\vec{U}_{in}}$$

$$S_i \frac{\vec{U}_i^{n+1} - \vec{U}_i^n}{\Delta t} = - \sum_{T \in \cup \Delta_i} \mathbb{B}_i^T \Phi^T(\vec{U}^n)$$

Lax-Wendroff : $\mathbb{B}_j^T = \frac{1}{3}\mathbb{I} + \frac{\Delta t}{2 S_T} \mathbb{K}_j^T$

LDA : $\mathbb{B}_j^T = \mathbb{K}_j^+ \left(\sum_{j \in T} \mathbb{K}_j^+ \right)^{-1}$

- Implicit mass-matrix (Rosiello et al 2005)
- Explicit High-Order Mass-Lumping (Ricchiutto & Abgrall 2010)

Implicit Consistent Mass-Matrix

$$\sum_{T \in \cup \Delta_i} \left[\sum_{j \in T} \mathbb{M}_{ij}^{T} \frac{\vec{U}_j^{n+1} - \vec{U}_j^{n}}{\Delta t} + \frac{1}{2} \left(\mathbb{B}_i^T \Phi^T \left(\vec{U}^{n+1} \right) + \mathbb{B}_i^T \Phi^T \left(\vec{U}^{n} \right) \right) \right] = 0$$

$$\mathbb{M}_{ij}^{T} = \frac{S_T}{3} \begin{pmatrix} \mathbb{B}_{j=1}^T & \mathbb{B}_{j=1}^T & \mathbb{B}_{j=1}^T \\ \mathbb{B}_{j=2}^T & \mathbb{B}_{j=2}^T & \mathbb{B}_{j=2}^T \\ \mathbb{B}_{j=3}^T & \mathbb{B}_{j=3}^T & \mathbb{B}_{j=3}^T \end{pmatrix}$$

Simple-Upwind Mass-Matrix

Solve for $\vec{U}_j^{n+1}$ using **Jacobi's iteration**.

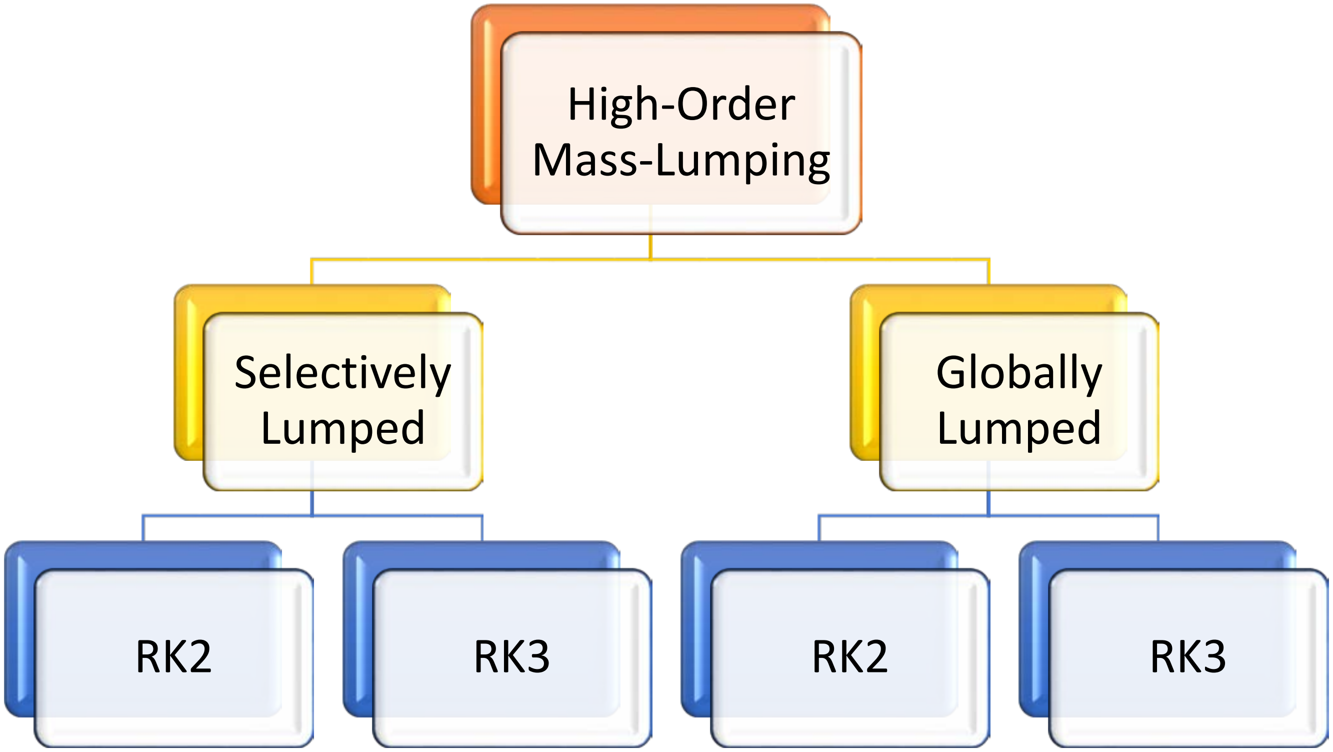
$$\sum_{T \in \cup \Delta_i} \left[\mathbb{M}_{ii}^{T} + \frac{1}{2} \mathbb{B}_i^T \mathbb{K}_i^T \right] \vec{U}_i^{n+1} = - \sum_{T \in \cup \Delta_i} \left[\sum_{\substack{j \in T \\ j \neq i}} \mathbb{M}_{ij}^{T} \frac{\vec{U}_j^{n+1}}{\Delta t} + \frac{1}{2} \mathbb{B}_i^T \sum_{\substack{j \in T \\ j \neq i}} \mathbb{K}_j^T \vec{U}_j^{n+1} \right] - \sum_{T \in \cup \Delta_i} \left[- \sum_{j \in T} \mathbb{M}_{ij}^{T} \frac{\vec{U}_j^{n}}{\Delta t} + \frac{1}{2} \mathbb{B}_i^T \Phi^T \left(\vec{U}^{n} \right) \right]$$

Iteration : $k+1$

Iteration : k

Iteration : k

Explicit High-Order Mass-Lumping



Selectively-Lumped (two-stage RK)

Stage 1:
$$S_i \frac{\vec{U}_i^1 - \vec{U}_i^n}{\Delta t} = - \sum_{T \in \cup \Delta_i} \mathbb{B}_i^T \Phi^T\left(\vec{U}^n\right)$$

$$\Phi^T\left(\vec{U}^n\right) = \sum_{j \in T} \mathbb{K}_j \vec{U}_j^n$$

Stage 2:
$$S_i \frac{\vec{U}_i^{n+1} - \vec{U}_i^n}{\Delta t} = - \sum_{T \in \cup \Delta_i} \left[\sum_{j \in T} \left(\mathbb{M}_{ij}^T - \mathbb{M}_{ij}^G\right) \frac{\vec{U}_j^1 - \vec{U}_j^n}{\Delta t} + \frac{1}{2}\left(\mathbb{B}_i^T \Phi^T\left(\vec{U}^1\right) + \mathbb{B}_i^T \Phi^T\left(\vec{U}^n\right)\right) \right]$$

$$\mathbb{M}_{ij}^G = \frac{S_T}{12}\left(\delta_{ij}\mathbb{I} + \mathbb{1}\right)$$

The explicit RD framework of Ricchiutto and Abgrall could be summarised in three steps:

Step 1 : Bubble Stabilization

The weak formulation of the upwind RD scheme

$$\iint_T \omega_i \left(\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{\mathbb{F}}\right) dx\, dy = 0$$

$$\iint_T (\psi_i + \gamma_i) \left(\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{\mathbb{F}}\right) dx\, dy = 0$$

Lagrange's basis function (Galerkin's approach if used alone)

$$\omega_i = \psi_i + \gamma_i$$

Step 2 : Time-Shifted Stabilization Operator

$$\iint_T \psi_i \left(\frac{\delta \vec{U}^k}{\Delta t} + \nabla \cdot \vec{\mathbb{F}} \right) dx\, dy + \iint_T \gamma_i \left(\frac{\overline{\delta \vec{U}^k}}{\Delta t} + \nabla \cdot \vec{\mathbb{F}} \right) dx\, dy = 0$$

where for RK2

$$\delta \vec{U}^1 = \vec{U}^1 - \vec{U}^n$$
$$\delta \vec{U}^2 = \vec{U}^2 - \vec{U}^n$$

$$\overline{\delta \vec{U}^1} = 0$$
$$\overline{\delta \vec{U}^2} = \vec{U}^1 - \vec{U}^n$$

$$\iint_T \psi_i \frac{\delta \vec{U}_i^k}{\Delta t} dx\, dy - \iint_T \psi_i \frac{\overline{\delta \vec{U}_i^k}}{\Delta t} dx\, dy = - \sum_{T \in \cup \Delta_i} \Phi_i^{RK(k)}$$

Step 3 : High-Order Mass-Lumping

$$\overline{\mathbb{M}}_{ij}^T = \mathbb{M}_{ij}^T + K \iint_T (\psi_i - \bar{\psi}_i)\psi_i \, dx\, dy$$

does not affect spatial accuracy

$$= \iint_T \omega_i \psi_i \, dx\, dy + K \iint_T (\psi_i - \bar{\psi}_i)\psi_i \, dx\, dy$$

$$\delta \mathbb{M}_{ij}^T = \frac{S_T}{36} \left(3\delta_{ij}\mathbb{I} - \mathbb{1} \right)$$

Take $K = 3$

$$\overbrace{\frac{S_T}{12}\left(3\delta_{ij}\mathbb{I} - \mathbb{1}\right)}^{3\delta\mathbb{M}_{ij}^{T}} = \frac{S_T}{3}\delta_{ij}\mathbb{I} - \underbrace{\frac{S_T}{12}\left(\delta_{ij}\mathbb{I} + \mathbb{1}\right)}_{\text{Galerkin's}}$$

$$S_i \frac{\delta \vec{U}_i^k}{\Delta t} = -\sum_{T \in \cup \Delta_i} \left[\Phi_i^{RK(k)} - \iint_T \psi_i \frac{\overline{\delta \vec{U}_i^k}}{\Delta t} \, dx \, dy \right]$$

Test Cases : Rectangular Waveguide (TE mode)

Test Case : 2D Unsteady Maxwell's Equations (TE mode)

The complete solution of 3D Maxwell's equation (TE mode : $E_z = 0$)

$$H_z = H_0 \cos(\kappa_m x) \cos(\kappa_n x) \exp[j(\omega t - \beta_{mn} z)]$$

$$H_x = j\beta_{mn} \frac{\kappa_m}{\kappa_{mn}^2} H_0 \sin(\kappa_m x) \cos(\kappa_n x) \exp[j(\omega t - \beta_{mn} z)]$$

$$H_y = j\beta_{mn} \frac{\kappa_n}{\kappa_{mn}^2} H_0 \cos(\kappa_m x) \sin(\kappa_n x) \exp[j(\omega t - \beta_{mn} z)]$$

$$E_x = j\omega\mu \frac{\kappa_n}{\kappa_{mn}^2} H_0 \cos(\kappa_m x) \sin(\kappa_n x) \exp[j(\omega t - \beta_{mn} z)]$$

$$E_y = -j\omega\mu \frac{\kappa_m}{\kappa_{mn}^2} H_0 \sin(\kappa_m x) \cos(\kappa_n x) \exp[j(\omega t - \beta_{mn} z)]$$

indicating for the standing wave mode $\kappa_m = \frac{m\pi}{a}$ $\kappa_n = \frac{n\pi}{b}$

$$\kappa_{mn}^2 = \kappa_m^2 + \kappa_n^2$$

Propagation coefficient $\beta_{mn}^2 = k^2 - \kappa_{mn}^2 = \omega^2\mu\varepsilon - \kappa_{mn}^2$

$$H_x = 0 = j\beta_{mn}\frac{\kappa_m}{\kappa_{mn}^2}H_0\sin(\kappa_m x)\cos(\kappa_n x)\exp[j(\omega t - \beta_{mn}z)]$$

$$H_y = 0 = j\beta_{mn}\frac{\kappa_n}{\kappa_{mn}^2}H_0\cos(\kappa_m x)\sin(\kappa_n x)\exp[j(\omega t - \beta_{mn}z)]$$

$$\beta_{mn} = 0$$

$$H_z = H_0\cos(\kappa_m x)\cos(\kappa_n x)\cos(\omega t + \phi)$$

$$E_x = -\,\omega\mu\frac{\kappa_n}{\kappa_{mn}^2}H_0\cos(\kappa_m x)\sin(\kappa_n x)\sin(\omega t + \phi)$$

$$E_y = j\omega\mu\frac{\kappa_m}{\kappa_{mn}^2}H_0\sin(\kappa_m x)\cos(\kappa_n x)\sin(\omega t + \phi)$$

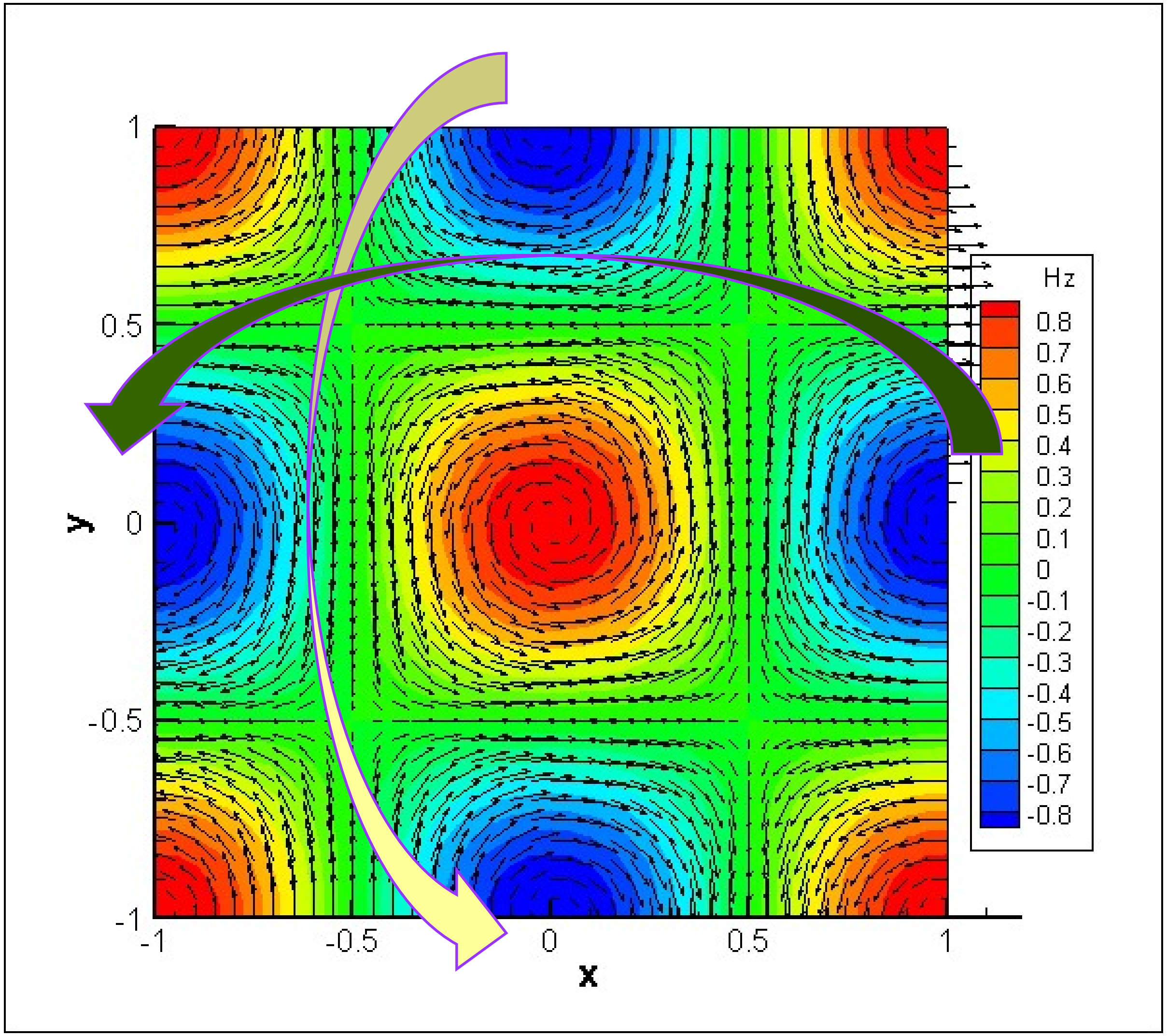
Introducing the phase difference ϕ into the solutions, without affecting the results.

and the **angular frequency** is given as

$$\beta_{mn}^2 = 0 \qquad \Rightarrow \qquad \omega^2\mu\varepsilon = \kappa_{mn}^2$$

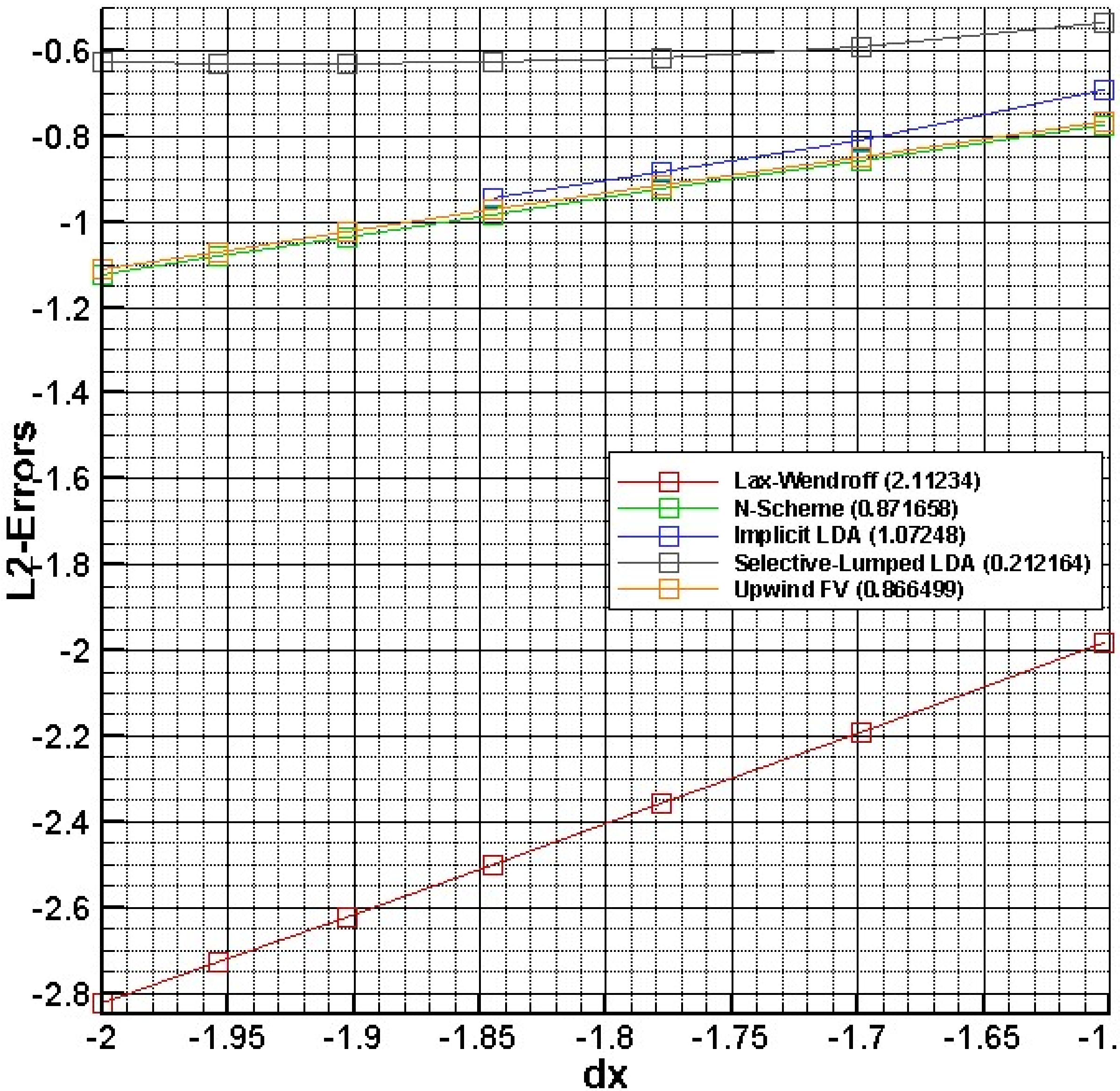
$$\omega = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

is known as the **cut-off frequency** of the rectangular waveguide



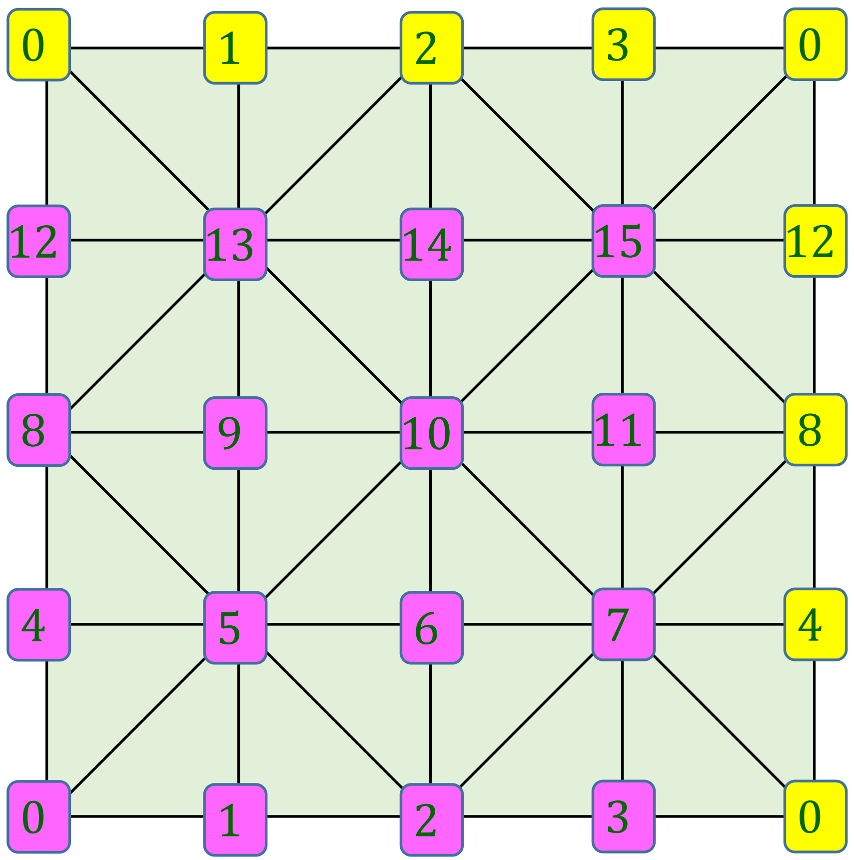
$m = 2$ and $n = 2$ have to be *even* integer in order to apply the periodic boundary condition.

L_2 - Errors

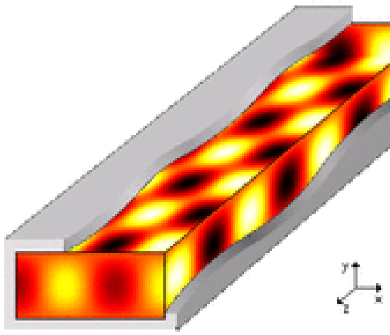


L2-Errors
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
-2
-2.2
-2.4
-2.6
-2.8
-2
-1.95
-1.9
-1.85
-1.8
-1.75
-1.7
-1.65
dx
Lax-Wendroff (2.11234)
N-Scheme (0.871658)
Implicit LDA (1.07248)
Selective-Lumped LDA (0.212164)
Upwind FV (0.866499)

Periodic Boundary Condition

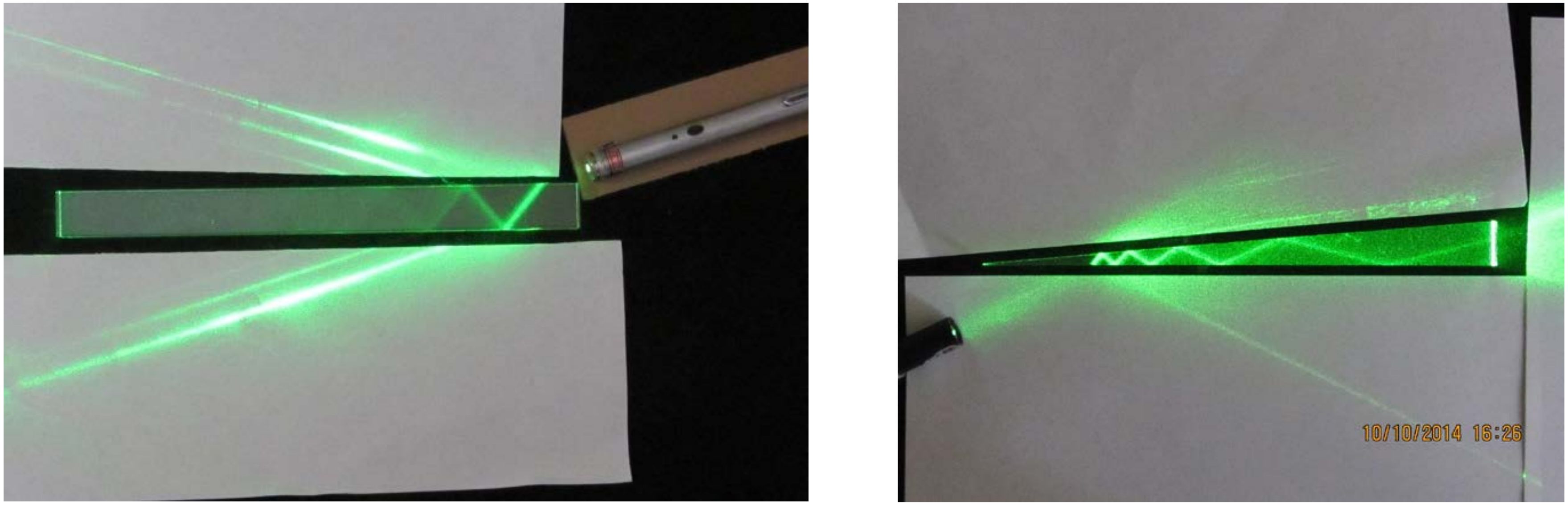


Examples of Rectangular Waveguides

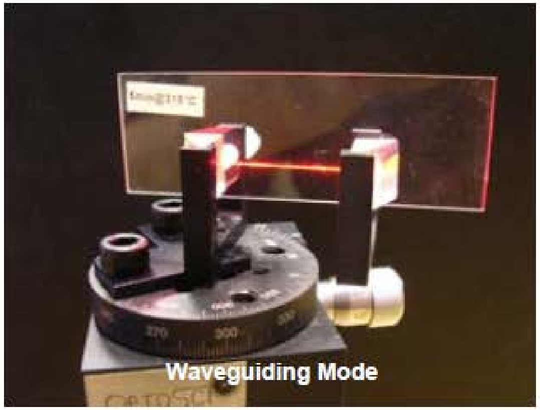


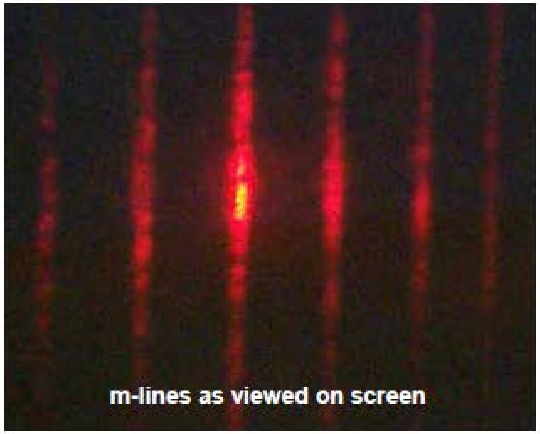
Waveguide propagation

Rectangular waveguide



Example of waveguide experiments





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