2D "Steady" State Test Case for Maxwell's Equations

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Maxwell's Equations

- 3D – Unsteady
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Maxwell's Equations in 3D

The last two coupled Maxwell's equations for electrodynamics are given as

$$\mu \frac{\partial \vec{H}}{\partial t} + \nabla \times \vec{E} = 0$$

$$\varepsilon \frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{H} = \vec{J}$$

Assuming there is no source current, $\vec{J} = 0$

$$\mu \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}$$

$$\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}$$

$$\mu \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}$$

$$\varepsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}$$

$$\varepsilon \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}$$

$$\varepsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}$$

Maxwell's Equations in 2D

Transverse Electric Wave (TE)

$E_z = 0$ and we further assume that $H_x = H_y = 0$

$$\mu \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}$$

$$\varepsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y}$$

$$\varepsilon \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}$$

Transverse Magnetic Wave (TM)

$H_z = 0$ and we further assume that $E_x = E_y = 0$

$$\mu \frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\partial y}$$

$$\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x}$$

$$\varepsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}$$

Maxwell's Equations in 1D

Transverse Electric Wave (TE)

$E_z = 0$ and we further assume that $H_x = H_y = E_x = 0$

$$\mu \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x}$$

$$\varepsilon \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}$$

Transverse Magnetic Wave (TM)

$H_z = 0$ and we further assume that $E_x = E_y = H_x = 0$

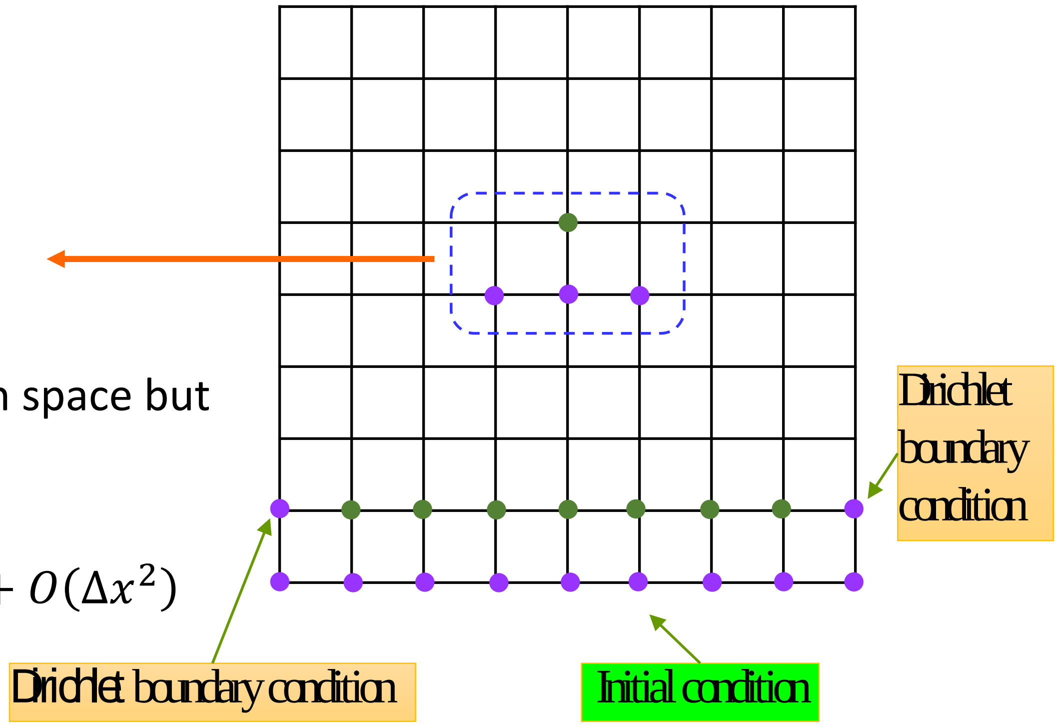
$$\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x}$$

$$\varepsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x}$$

Let us discretise the linear scalar advection problem using *Forward-Time Centered-Space* finite-difference method:

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\,\Delta x} \cong 0$$



The scheme is second-order-accurate in space but first-order-accurate in time.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\,\Delta x} = O(\Delta t) + O(\Delta x^2)$$

Consider for TE-mode in 1-D,

$$\mu \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x}$$

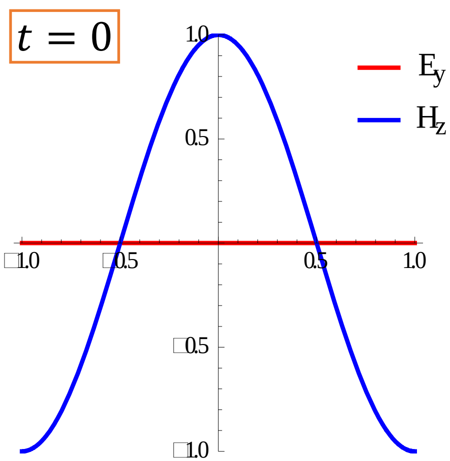
$$\varepsilon \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}$$

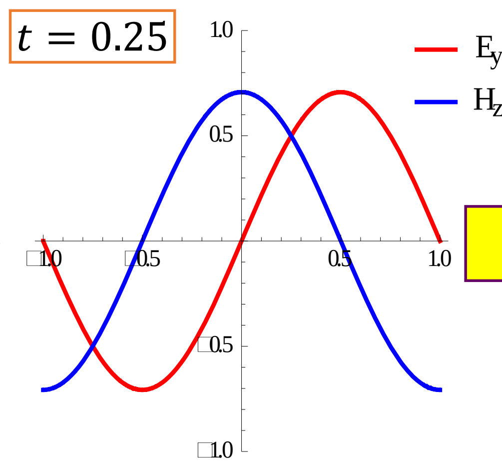
For mode $m = 1$

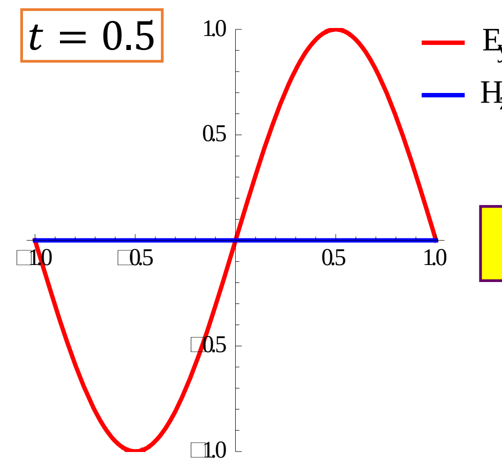
The analytical solution for the TE mode problem is

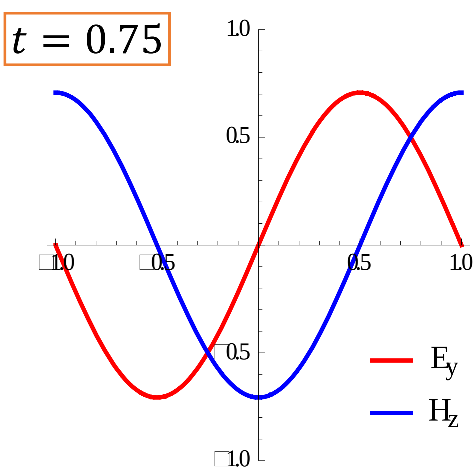
$$H_{zm}(x,t) = B \cos \omega_m t \cos \frac{m\pi x}{L}$$

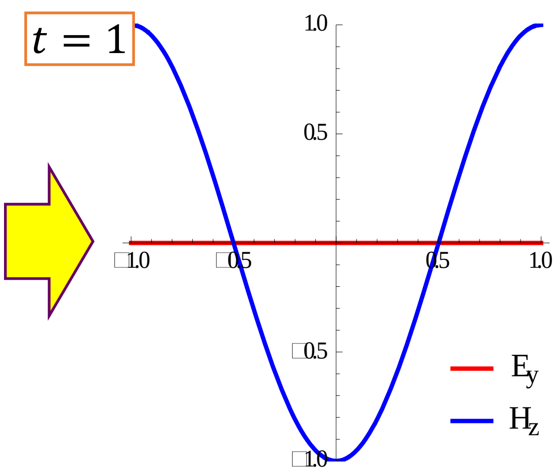
$$E_{ym}(x,t) = \frac{B}{\varepsilon \omega_m} \frac{m\pi}{L} \sin \omega_m t \sin \frac{m\pi x}{L}$$

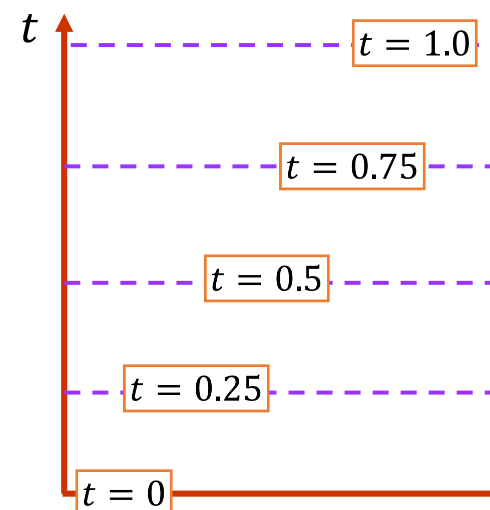












Unfortunately, this is still an unsteady problem.

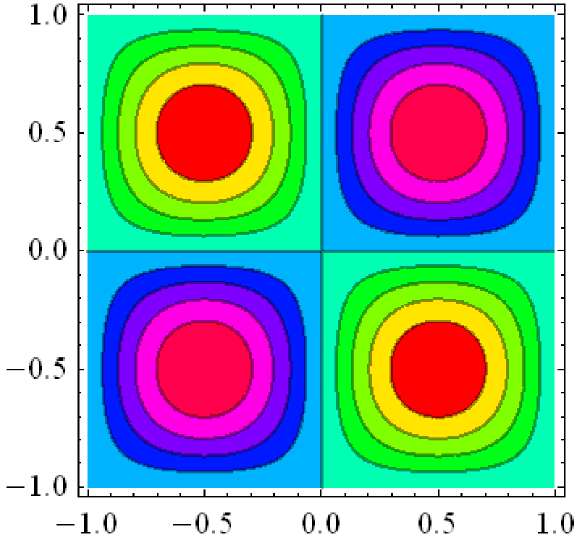
1D Unsteady problem → 2D Steady Problem

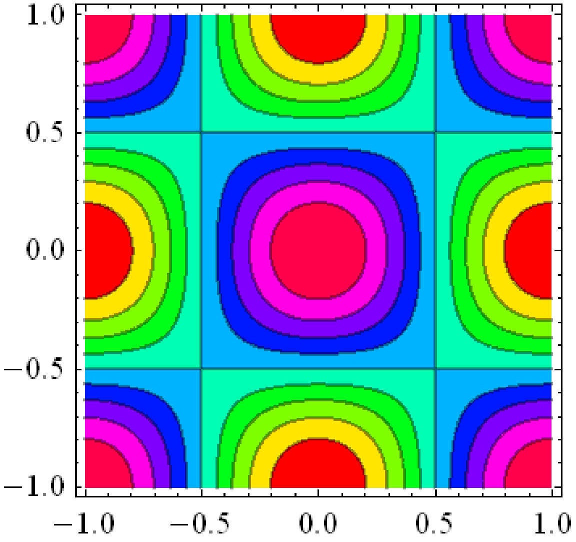
We can map $t \rightarrow y$

$$H_{zm}(x,t) = B\cos\omega_m t\cos\frac{m\pi x}{L} \quad \Longrightarrow \quad H_{zm}(x,y) = B\cos\omega_m y\cos\frac{m\pi x}{L}$$

$$E_{ym}(x,t) = \frac{B}{\varepsilon\omega_m}\frac{m\pi}{L}\sin\omega_m t\sin\frac{m\pi x}{L} \quad \Longrightarrow \quad E_{ym}(x,y) = \frac{B}{\varepsilon\omega_m}\frac{m\pi}{L}\sin\omega_m y\sin\frac{m\pi x}{L}$$

$m = 1$



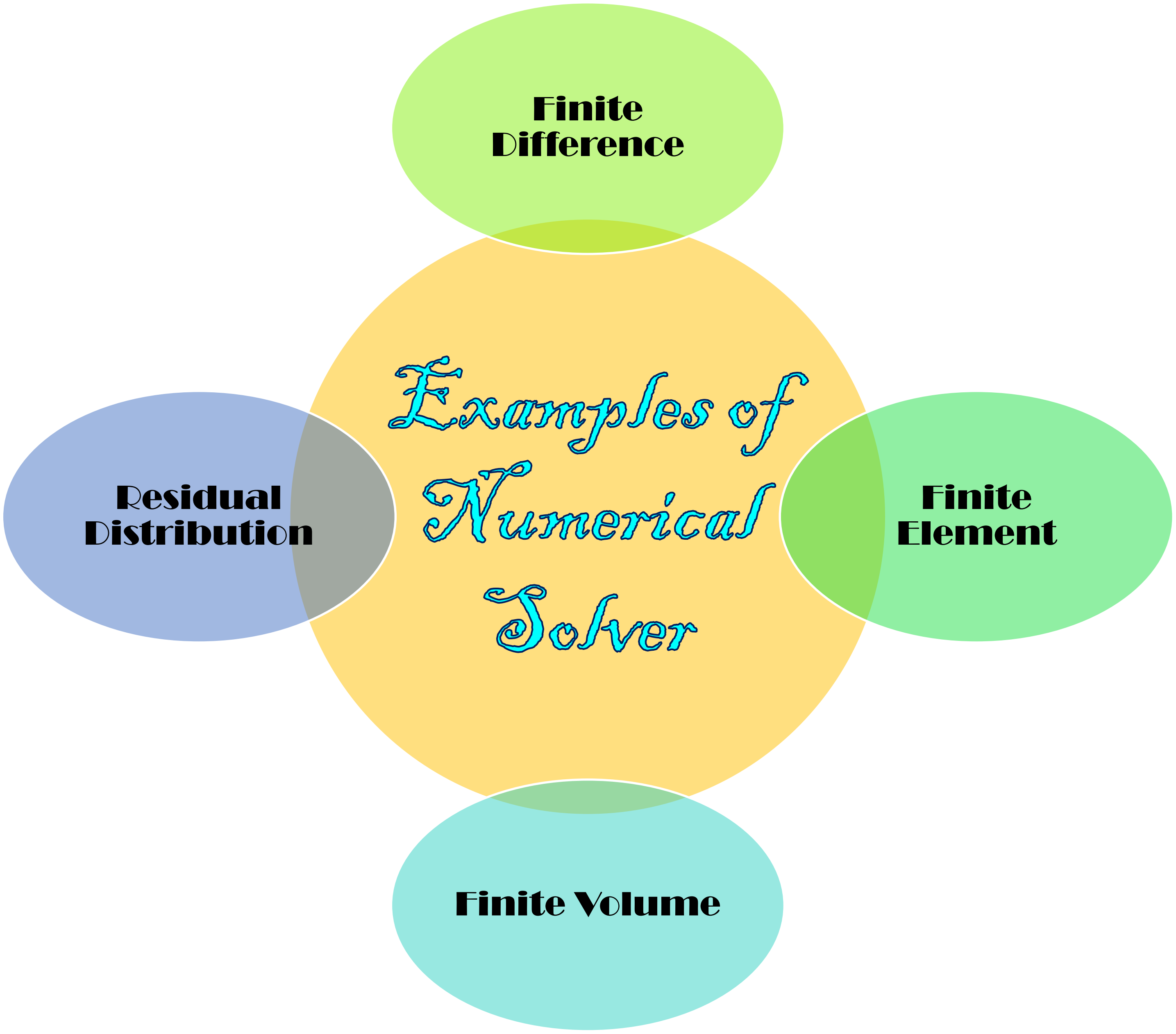


According to the physical significance of the problem,

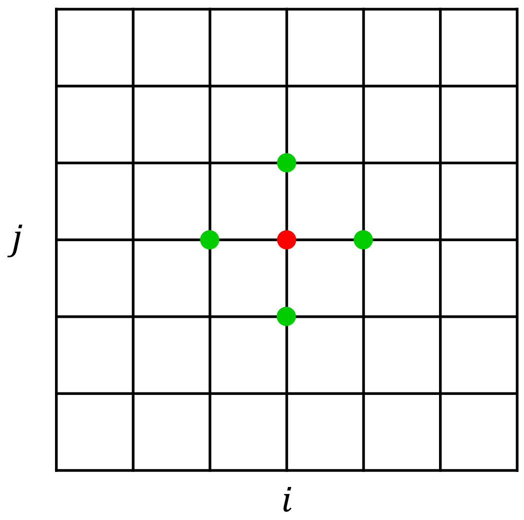
m-mode angular frequency, $\omega_m = \frac{mc\pi}{L}$ m is the mode number

Brief Descriptions of Numerical Solvers

- Explanations of Numerical Techniques
- Pseudo-time Iteration


Examples of Numerical Solver
Finite Difference
Residual Distribution
Finite Element
Finite Volume

Finite-Difference

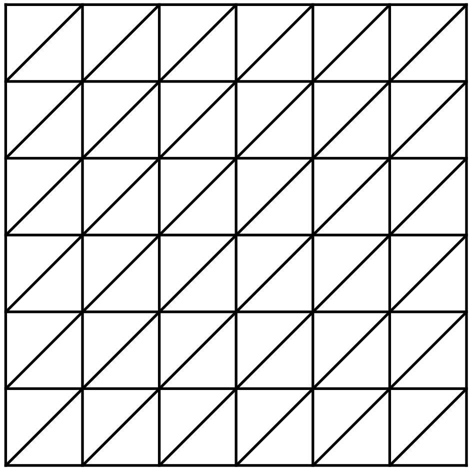


To discretise the coupled first-order equations to be in second-order-accurate, $(\Delta x^2, \Delta y^2)$

$$\frac{\partial E_y}{\partial x} + \mu \frac{\partial H_z}{\partial y} = 0 \quad \Rightarrow \quad \frac{E_y[i+1,j] - E_y[i-1,j]}{2\,\Delta x} + \mu \frac{H_z[i,j+1] - H_z[i,j-1]}{2\,\Delta y} = 0$$

$$\frac{\partial H_z}{\partial x} + \varepsilon \frac{\partial E_y}{\partial y} = 0 \quad \Rightarrow \quad \frac{H_z[i+1,j] - H_z[i-1,j]}{2\,\Delta x} + \varepsilon \frac{E_y[i,j+1] - E_y[i,j-1]}{2\,\Delta y} = 0$$

Finite-Element



i Multiplying the Equations with weight function, $w = w(x,y)$

$$w\frac{\partial H_z}{\partial x} + \varepsilon w\frac{\partial E_y}{\partial y} = 0$$

$$w\frac{\partial E_y}{\partial x} + \mu w\frac{\partial H_z}{\partial y} = 0$$

ii Integrate over the control volume

$$\int_T \left(w\frac{\partial H_z}{\partial x} + \varepsilon w\frac{\partial E_y}{\partial y}\right) dxdy = 0$$

$$\int_T \left(w\frac{\partial E_y}{\partial x} + \mu w\frac{\partial H_z}{\partial y}\right) dxdy = 0$$

iii Weak formulation (integration by parts from step (ii))

$$\oint_{\partial T} w(H_z\,dy - \varepsilon E_y\,dx) - \int_T \left(H_z\frac{\partial w}{\partial x} + \varepsilon E_y\frac{\partial w}{\partial y}\right) dxdy = 0 \Rightarrow \int_T \left(H_z\frac{\partial w}{\partial x} + \varepsilon E_y\frac{\partial w}{\partial y}\right) dxdy = 0$$

$$\oint_{\partial T} w(E_y dy - \mu H_z dx) - \int_T \left(E_y\frac{\partial w}{\partial x} + \mu H_z\frac{\partial w}{\partial y}\right) dxdy = 0 \Rightarrow \int_T \left(E_y\frac{\partial w}{\partial x} + \mu H_z\frac{\partial w}{\partial y}\right) dxdy = 0$$

iv Conserved variables as interpolating basis function

$$E_{iy}(x,y) = E_{iy}\psi_i(x,y)$$

$$H_{iz}(x,y) = H_{iz}\psi_i(x,y)$$

$$H_{iz}\int_T w\frac{\partial \psi_i}{\partial x} dxdy + E_{iy}\int_T \varepsilon w\frac{\partial \psi_i}{\partial y} dxdy = 0$$

$$E_{iy}\int_T w\frac{\partial \psi_i}{\partial x} dxdy + H_{iz}\int_T \mu w\frac{\partial \psi_i}{\partial y} dxdy = 0$$

v Galerkin's approach : weight function same as the basis function $w(x,y) = \psi_j(x,y)$

$$H_{iz}\int_T \psi_j\frac{\partial \psi_i}{\partial x} dxdy + E_{iy}\int_T \varepsilon\psi_j\frac{\partial \psi_i}{\partial y} dxdy = 0$$

$$E_{iy}\int_T \psi_j\frac{\partial \psi_i}{\partial x} dxdy + H_{iz}\int_T \mu\psi_j\frac{\partial \psi_i}{\partial y} dxdy = 0$$

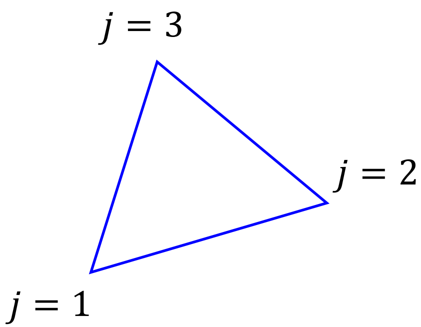
Example of Weak Formulation for Single Variable:

$$\int_T \left(w\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} \right) dxdy = \int_T (wu\,\hat{x} + wu\,\hat{y}) \cdot \vec{n}\, dl - \int_T \left(u\frac{\partial w}{\partial x} + u\frac{\partial w}{\partial y} \right) dxdy$$

integration by parts

lower the order of differential equation by 1

The gradients of the local basis functions are always <u>constant</u> within each element.



$$H_{iz}\frac{\partial \psi_i}{\partial x}\int_T \psi_j dxdy + \varepsilon E_{iy}\frac{\partial \psi_i}{\partial y}\int_T \psi_j dxdy = 0$$

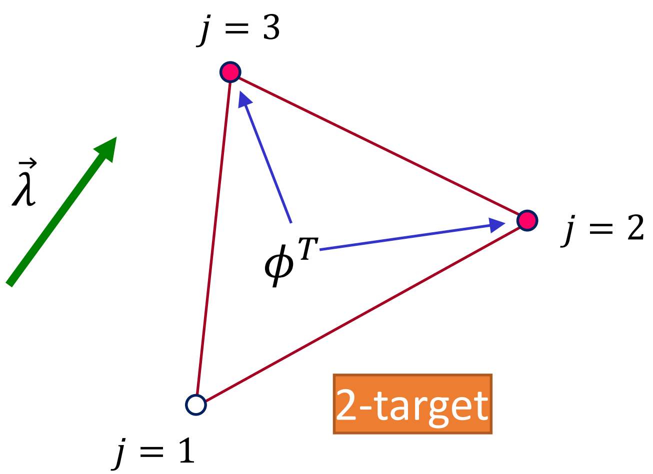
$$E_{iy}\frac{\partial \psi_i}{\partial x}\int_T \psi_j dxdy + \mu H_{iz}\frac{\partial \psi_i}{\partial y}\int_T \psi_j dxdy = 0$$

Local stiffness matrix

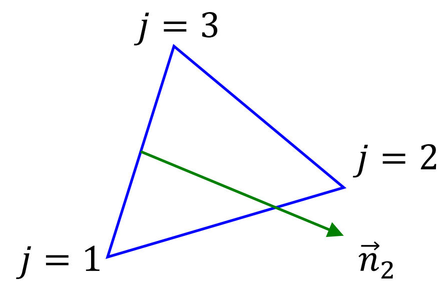
$$\frac{1}{6}\begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \end{bmatrix} \begin{Bmatrix} H_{1z} \\ H_{2z} \\ H_{3z} \end{Bmatrix} + \frac{\varepsilon}{6}\begin{bmatrix} (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} E_{1y} \\ E_{2y} \\ E_{3y} \end{Bmatrix} = 0$$

$$\frac{1}{6}\begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \end{bmatrix} \begin{Bmatrix} E_{1y} \\ E_{2y} \\ E_{3y} \end{Bmatrix} + \frac{\mu}{6}\begin{bmatrix} (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} H_{1z} \\ H_{2z} \\ H_{3z} \end{Bmatrix} = 0$$

Residual Distribution



Step 1 : Calculate flux residual $\phi^T = \oint_{\partial T} \vec{\mathcal{F}} \cdot \vec{n}\, dl \cong \frac{1}{2} \sum_{j \in T} \vec{\mathcal{F}} \cdot \vec{n}_j$



Step 2 : Distribute the residual $\phi_j^T = \beta_j^T \phi^T$

2-target		
$\phi_1^T = 0$	$\phi_2^T = \beta_2^T \phi^T$	$\phi_3^T = \beta_3^T \phi^T$

1-target		
$\phi_1^T = 0$	$\phi_2^T = 0$	$\phi_3^T = \beta_3^T \phi^T$

Step 3 : Nodal update

$$u_i^{n+1} = u_i^n - \frac{\Delta\tau}{S_i} \sum_{T \in \cup \Delta_i} \beta_i^T \phi^T$$

The "steady" state governing equation is given by

$$\frac{\partial H_z}{\partial x} + \varepsilon \frac{\partial E_y}{\partial y} = 0$$

$$\frac{\partial E_y}{\partial x} + \mu \frac{\partial H_z}{\partial y} = 0$$

The "steady" state problem in RD is usually solved using pseudo-time iteration.

$$\left.\begin{aligned}\frac{\partial E_y}{\partial \tau} + \frac{\partial H_z}{\partial x} + \varepsilon \frac{\partial E_y}{\partial y} = 0 \\ \frac{\partial H_z}{\partial \tau} + \frac{\partial E_y}{\partial x} + \mu \frac{\partial H_z}{\partial y} = 0\end{aligned}\right]$$

$$\frac{\partial}{\partial \tau}\underbrace{\begin{pmatrix} E_y \\ H_z \end{pmatrix}}_{\vec{U}} + \frac{\partial}{\partial x}\underbrace{\begin{pmatrix} H_z \\ E_y \end{pmatrix}}_{\vec{F}} + \frac{\partial}{\partial y}\underbrace{\begin{pmatrix} \varepsilon E_y \\ \mu H_z \end{pmatrix}}_{\vec{G}} = 0$$

$\vec{U}$: conserved variables; $\vec{F}$: fluxes; $\vec{G}$: fluxes

Let's focus on **LDA scheme** for hyperbolic system only. The distribution matrix is

$$\mathbb{B}_j^T = \frac{\mathbb{K}_j^+}{\sum_{j \in T} \mathbb{K}_j^+}$$

where the inflow matrix is given as

$$\mathbb{K}_j = \frac{1}{2}\left(n_{jx} \frac{\partial \vec{F}(\vec{U})}{\partial \vec{U}} + n_{jy} \frac{\partial \vec{G}(\vec{U})}{\partial \vec{U}} \right) = \frac{1}{2}\begin{pmatrix} \varepsilon n_{jy} & n_{jx} \\ n_{jx} & \mu n_{jy} \end{pmatrix}$$

The inflow matrix has to be determined whether in positive or negative

$$\mathbb{K}_j^{\pm} = \frac{-2\delta n_{jx}}{2}\begin{pmatrix} n_{jx} & n_{jx} \\ \lambda_1 - \varepsilon n_{jy} & \lambda_2 - \varepsilon n_{jy} \end{pmatrix}\begin{pmatrix} \lambda_1^{\pm} & 0 \\ 0 & \lambda_2^{\pm} \end{pmatrix}\begin{pmatrix} \lambda_2 - \varepsilon n_{jy} & -n_{jx} \\ \varepsilon n_{jy} - \lambda_1 & n_{jx} \end{pmatrix}$$

with the eigenvalues given as

$$\lambda_1 = \frac{\left(\mu n_{jy} + \varepsilon n_{jy}\right)}{2} - \delta$$

$$\lambda_2 = \frac{\left(\mu n_{jy} + \varepsilon n_{jy}\right)}{2} + \delta$$

$$\delta = \frac{\sqrt{\left(\mu n_{jy} - \varepsilon n_{jy}\right)^2 + 4{n_{jx}}^2}}{2}$$

The flux residual

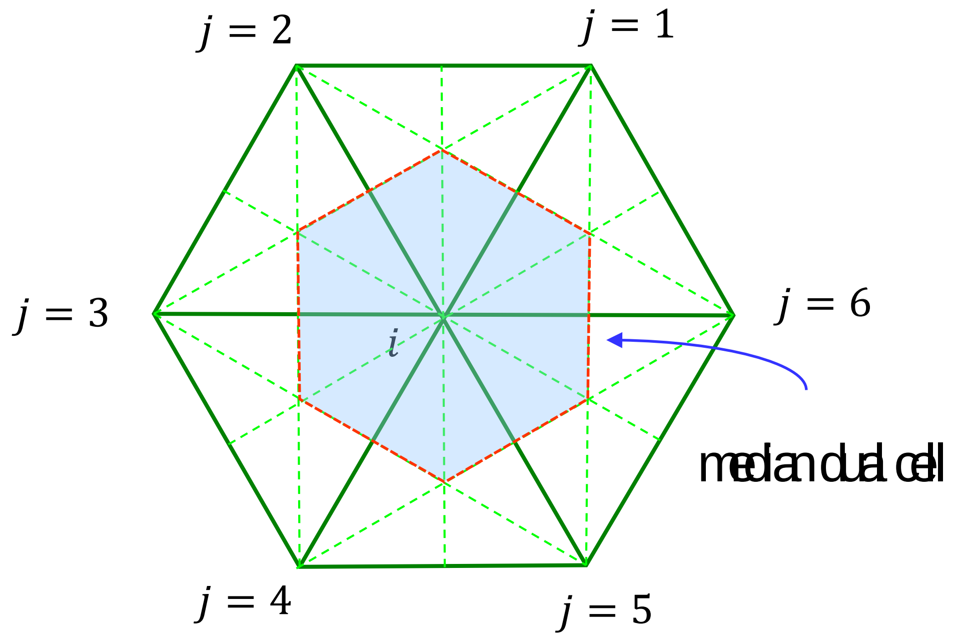
$$\Phi^T = \frac{1}{2}\sum_{j\in T}\vec{\mathcal{F}}\cdot\vec{n}_j = \frac{1}{2}\sum_{j\in T}\left(n_{jx}\vec{F}(\vec{U}) + n_{jy}\vec{G}(\vec{U})\right)$$

$$\vec{\mathcal{F}}(\vec{U}) = \left(\vec{F}(\vec{U}), \vec{G}(\vec{U})\right) \qquad \vec{F}(\vec{U}) = \begin{pmatrix} H_z \\ E_y \end{pmatrix} \qquad \vec{G}(\vec{U}) = \begin{pmatrix} \varepsilon E_y \\ \mu H_z \end{pmatrix}$$

Finally, the nodal update as

$$\vec{U}_i^{n+1} = \vec{U}_i^n - \frac{\Delta\tau}{S_i}\sum_{T\in\cup\Delta_i}\mathbb{B}_i^T\Phi^T$$

Finite-Volume



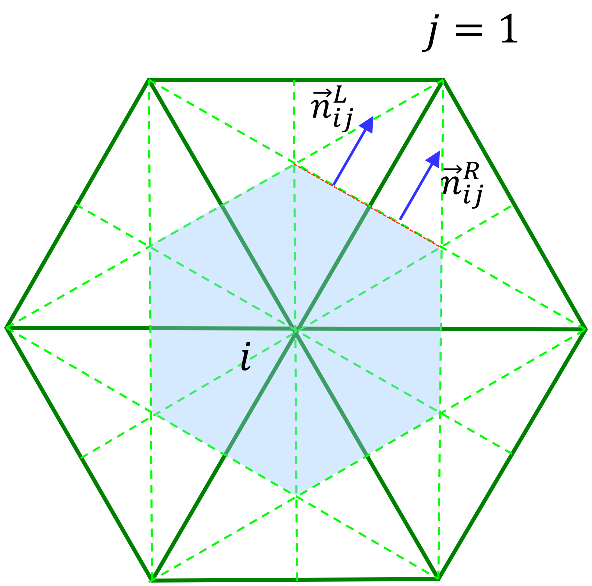
The semi-discrete equation is given as

$$S_i \frac{d\vec{U}_i}{d\tau} + \sum_{j \in \cup k_i} \left[\vec{H}\left(u_i, u_j, n_{ij}^L\right) + \vec{H}\left(u_i, u_j, n_{ij}^R\right) \right] = 0$$

where the numerical flux vectors are given as $\vec{\mathcal{F}} = (\vec{F}, \vec{G})$

$\vec{n}^{L}_{ij}$
$$\vec{H}(\vec{U}_i, \vec{U}_j, \vec{n}^{L}_{ij}) = \frac{\vec{\mathcal{F}}(\vec{U}_i)\cdot\vec{n}^{L}_{ij} + \vec{\mathcal{F}}(\vec{U}_j)\cdot\vec{n}^{L}_{ij}}{2} - \frac{1}{2}\left|\frac{\partial\vec{\mathcal{F}}}{\partial\vec{U}}\cdot\vec{n}^{L}_{ij}\right|(\vec{U}_j - \vec{U}_i)$$

$\vec{n}^{R}_{ij}$
$$\vec{H}(\vec{U}_i, \vec{U}_j, \vec{n}^{R}_{ij}) = \frac{\vec{\mathcal{F}}(\vec{U}_i)\cdot\vec{n}^{R}_{ij} + \vec{\mathcal{F}}(\vec{U}_j)\cdot\vec{n}^{R}_{ij}}{2} - \frac{1}{2}\left|\frac{\partial\vec{\mathcal{F}}}{\partial\vec{U}}\cdot\vec{n}^{R}_{ij}\right|(\vec{U}_j - \vec{U}_i)$$



This is a first-order-accurate upwind scheme. To accomplish the summation, one has to sum the numerical fluxes with all neighbouring nodes :

$$\partial S_i = \partial S_{i1} \cup \partial S_{i2} \cup \partial S_{i3} \cup \partial S_{i4} \cup \partial S_{i5} \cup \partial S_{i6}$$

Pseudo-time Iteration

Finite Difference

$$\frac{\partial E_{iy}}{\partial \tau} + \frac{E_y[i+1,j] - E_y[i-1,j]}{2\,\Delta x} + \mu \frac{H_z[i,j+1] - H_z[i,j-1]}{2\,\Delta y} = 0$$

$$\frac{\partial H_{iz}}{\partial \tau} + \frac{H_z[i+1,j] - H_z[i-1,j]}{2\,\Delta x} + \varepsilon \frac{E_y[i,j+1] - E_y[i,j-1]}{2\,\Delta x} = 0$$

Finite Volume

$$S_i \frac{d\vec{U}_i}{d\tau} + \sum_{j \in \cup k_i} \left[\vec{H}\left(u_i, u_j, n_{ij}^L\right) + \vec{H}\left(u_i, u_j, n_{ij}^R\right)\right] = 0$$

Finite Element

$$\frac{\partial E_{iy}}{\partial \tau} + \int_T \psi_i dxdy \sum_{T \in \cup \Delta_i} \sum_{j \in T} \left(H_{iz} \frac{\partial \psi_j}{\partial x} + \varepsilon E_{iy} \frac{\partial \psi_j}{\partial y}\right) = 0$$

$$\frac{\partial H_{iz}}{\partial \tau} + \int_T \psi_i dxdy \sum_{T \in \cup \Delta_i} \sum_{j \in T} \left(E_{iy} \frac{\partial \psi_j}{\partial x} + \mu H_{iz} \frac{\partial \psi_j}{\partial y}\right) = 0$$

Residual Distribution

$$\vec{U}_i^{n+1} = \vec{U}_i^n - \frac{\Delta \tau}{S_i} \sum_{T \in \cup \Delta_i} \mathbb{B}_i^T \Phi^T \qquad \Phi^T = \frac{1}{2} \sum_{j \in T} \left(n_{jx} \vec{F}(\vec{U}) + n_{jy} \vec{G}(\vec{U})\right)$$

$$\mathbb{B}_j^T = \frac{\mathbb{K}_j^+}{\sum_{j \in T} \mathbb{K}_j^+} \qquad \vec{F}(\vec{U}) = \begin{pmatrix} H_z \\ E_y \end{pmatrix} \qquad \vec{G}(\vec{U}) = \begin{pmatrix} \varepsilon E_y \\ \mu H_z \end{pmatrix}$$

- The reason for using pseudo-time iteration to solve all the four numerical methods is to allow a fair comparison in order-of-accuracy among them.
- (commonly seen for Finite Volume & Residual Distribution, but rarely used for Finite Difference or Finite Element)

Finite Difference	• **Second**-order-accurate • Discretisation in ***differential*** form • Central scheme
Finite Volume	• **First**-order-accurate • Discretisation in ***integral*** form • Upwind scheme
Finite Element	• **Second**-order-accurate • Discretisation in ***integral*** form • Galerkin's Central scheme
Residual Distribution	• **Second**-order-accurate • Discretisation in ***integral*** form • Upwind scheme (LDA) ; central scheme (Lax-Wendroff)

- The second-order finite-difference method is not diagonally dominant, it is not possible to be solved by matrix inversion.

$$\frac{\partial E_y}{\partial x} + \mu \frac{\partial H_z}{\partial y} = 0 \quad \Rightarrow \quad \frac{E_y[i+1,j] - E_y[i-1,j]}{2\,\Delta x} + \mu \frac{H_z[i,j+1] - H_z[i,j-1]}{2\,\Delta y} = 0$$

$$\frac{\partial H_z}{\partial x} + \varepsilon \frac{\partial E_y}{\partial y} = 0 \quad \Rightarrow \quad \frac{H_z[i+1,j] - H_z[i-1,j]}{2\,\Delta x} + \varepsilon \frac{E_y[i,j+1] - E_y[i,j-1]}{2\,\Delta x} = 0$$

$$\begin{bmatrix} 0 & \times & \times & \times & \times \\ \times & 0 & \times & \times & \times \\ \times & \times & 0 & \times & \times \\ \times & \times & \times & 0 & \times \\ \times & \times & \times & \times & 0 \end{bmatrix}$$

Numerical Results

- Test Case Description
- Initial Guess
- Contour Plot & Vector Plot
- L2-Errors

Numerical Test

The fundamental mode $m = 1$ is used for the test case.

The modified "steady" state governing equation reads

$$\frac{\partial E_y}{\partial x} + \mu \frac{\partial H_z}{\partial y} = 0$$
$$\frac{\partial H_z}{\partial x} + \varepsilon \frac{\partial E_y}{\partial y} = 0$$

The pseudo-time technique

$$\frac{\partial E_y}{\partial \tau} + \frac{\partial E_y}{\partial x} + \mu \frac{\partial H_z}{\partial y} = 0$$
$$\frac{\partial H_z}{\partial \tau} + \frac{\partial H_z}{\partial x} + \varepsilon \frac{\partial E_y}{\partial y} = 0$$

with the "steady" state analytical solution as

$$H_{zm}(x, y) = B \cos \omega_m y \cos \frac{m\pi x}{L}$$
$$E_{ym}(x, y) = \frac{B}{\varepsilon \omega_m} \frac{m\pi}{L} \sin \omega_m y \sin \frac{m\pi x}{L}$$

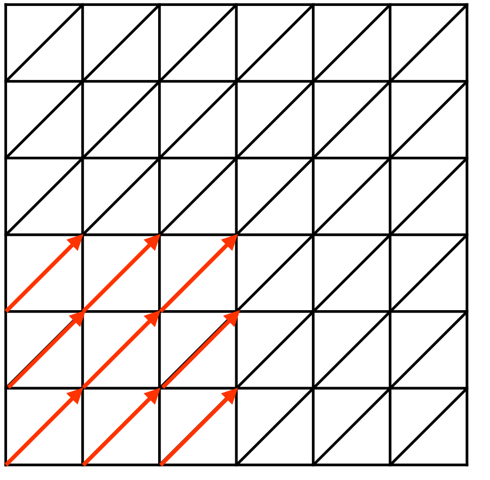
$$\omega_m = \frac{mc\pi}{L}$$

The permittivity and permeability are set to be

permeability $\mu = 1.1$

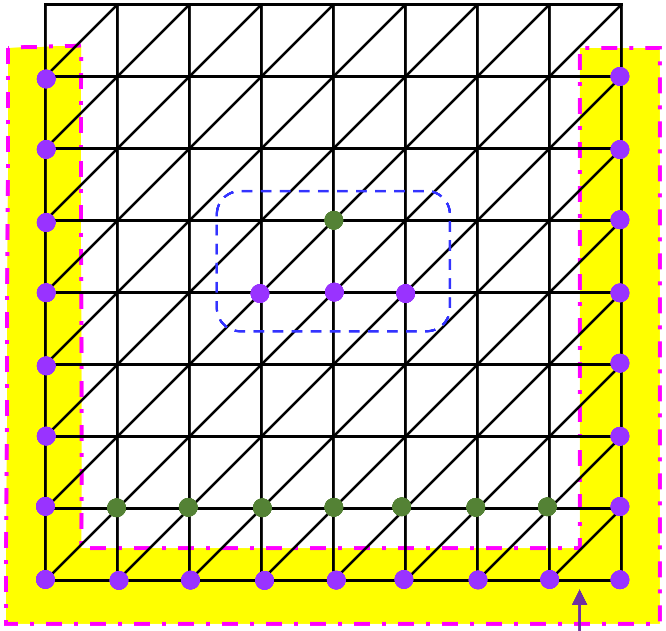
permittivity $\varepsilon = 1.1$

If $\mu = 1.0$ and $\varepsilon = 1.0$,
for RR grid with $\Delta x : \Delta y = 1 : 1$,
exact solution will be recovered.



Initial Guess

$$\frac{\partial E_y}{\partial y} + \frac{1}{\varepsilon}\frac{\partial H_z}{\partial x} = 0$$
$$\frac{\partial H_z}{\partial y} + \frac{1}{\mu}\frac{\partial E_y}{\partial x} = 0$$



Extending the discussions for 1D unsteady scalar advection problem to 2D unsteady problem (by changing t to y).

Due to the symmetry of the RR grid, the initial guesses could be generated using Lax-Wendroff Finite-Difference method (for coupled wave equation).

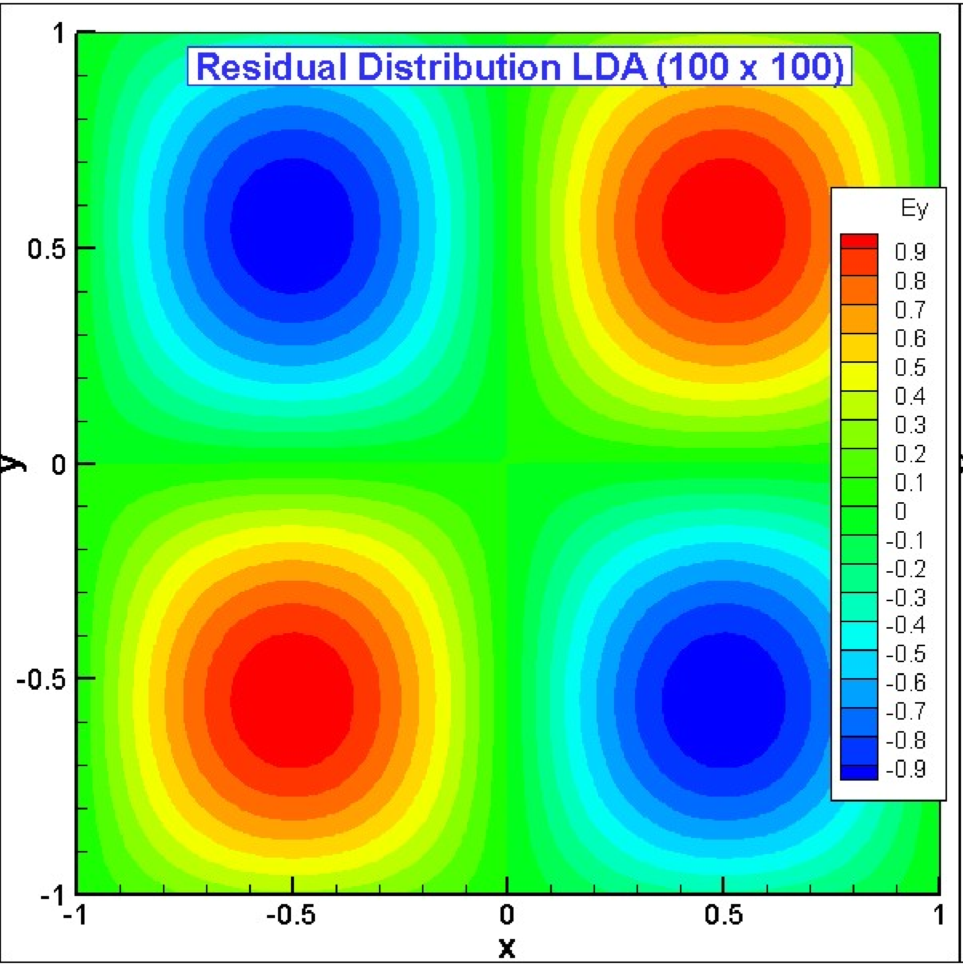
The boundary conditions needed at least to perform the second-order-accurate numerical scheme.

$$E_y[i,j+1] = E_y[i,j] - \frac{1}{2}\frac{\Delta t}{\varepsilon \Delta x}\left(H_z[i+1,j] - H_z[i-1,j]\right) + \frac{1}{2}\left(\frac{\Delta t}{\varepsilon \Delta x}\right)^2 \left(E_y[i+1,j] - 2E_y[i,j] + E_y[i-1,j]\right)$$

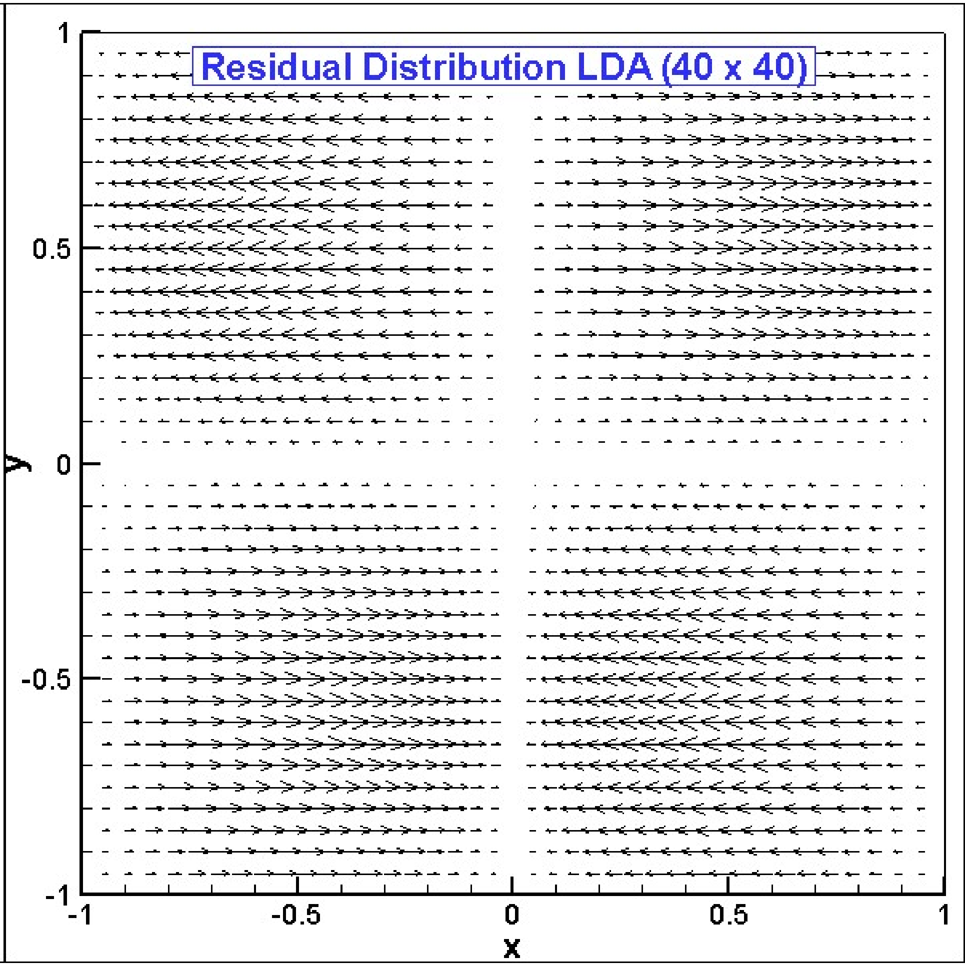
$$H_z[i,j+1] = H_z[i,j] - \frac{1}{2}\frac{\Delta t}{\varepsilon \Delta x}\left(E_y[i+1,j] - E_y[i-1,j]\right) + \frac{1}{2}\left(\frac{\Delta t}{\varepsilon \Delta x}\right)^2 \left(H_z[i+1,j] - 2H_z[i,j] + H_z[i-1,j]\right)$$

Contour & Vector Plot for LDA Scheme

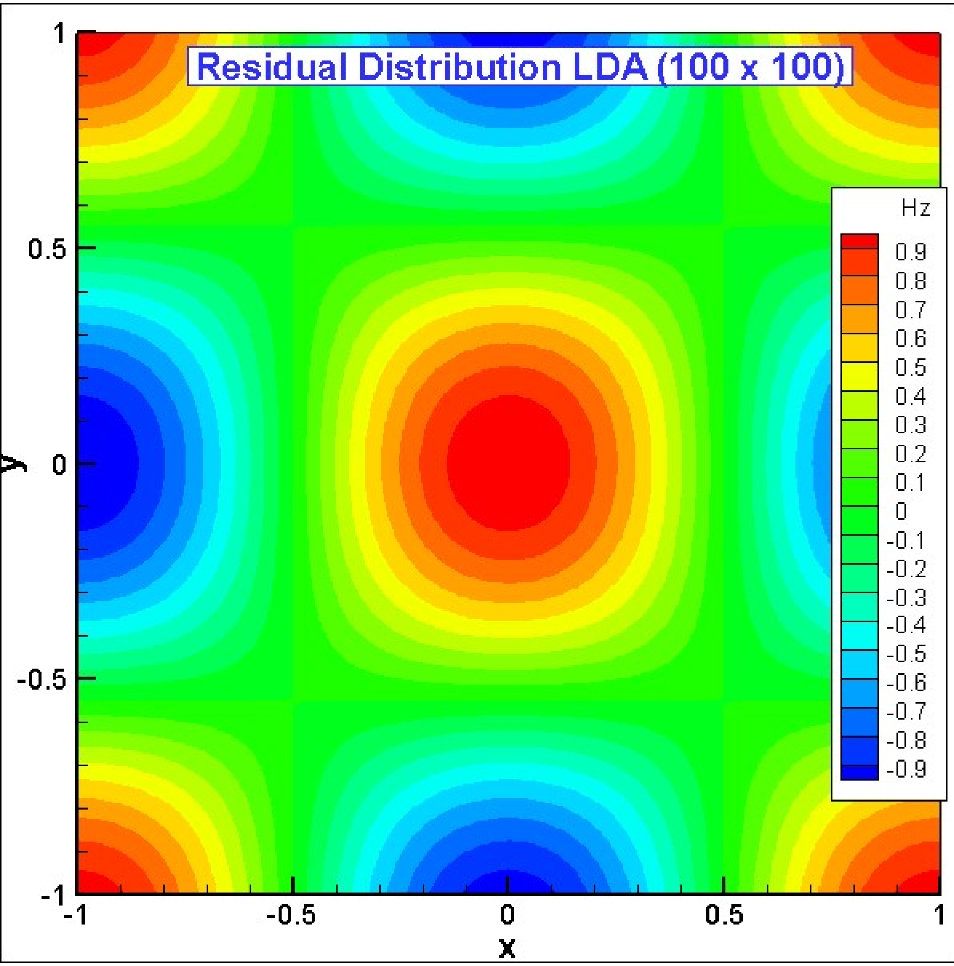
Contour Plot for E_y



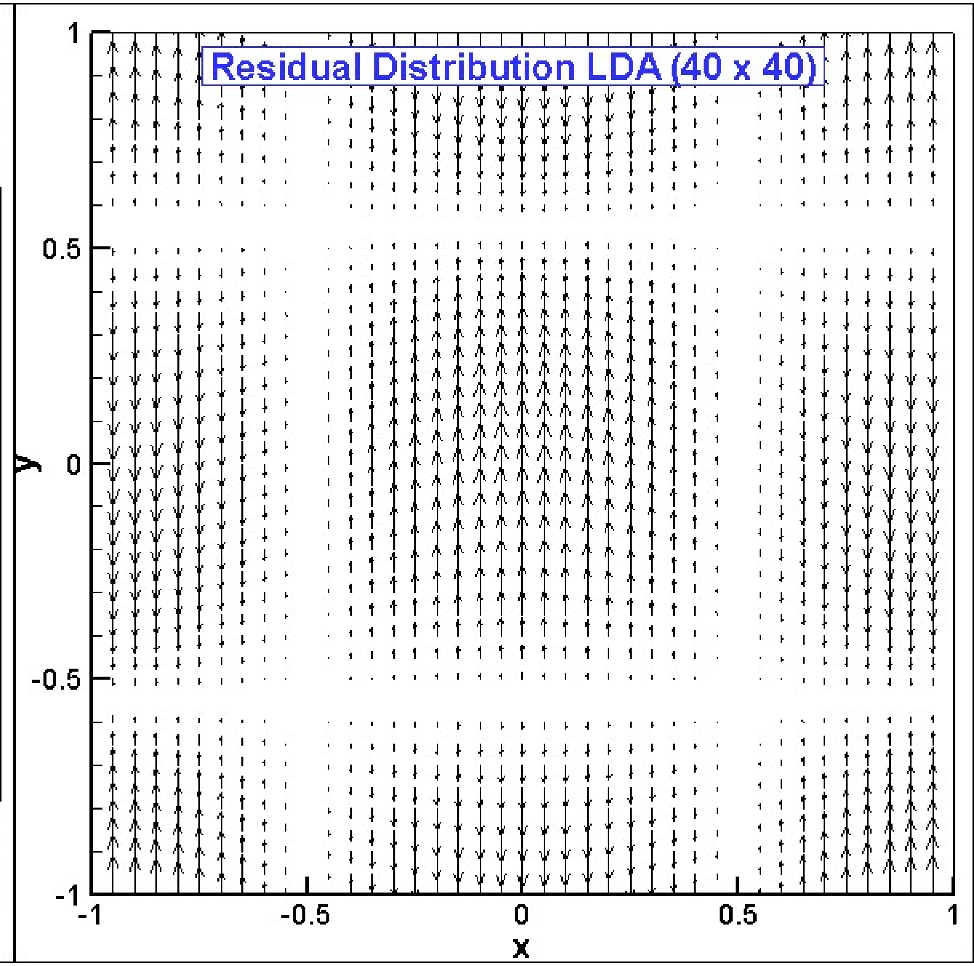
Vector Plot for (E_y, E_z)



Contour Plot for H_z

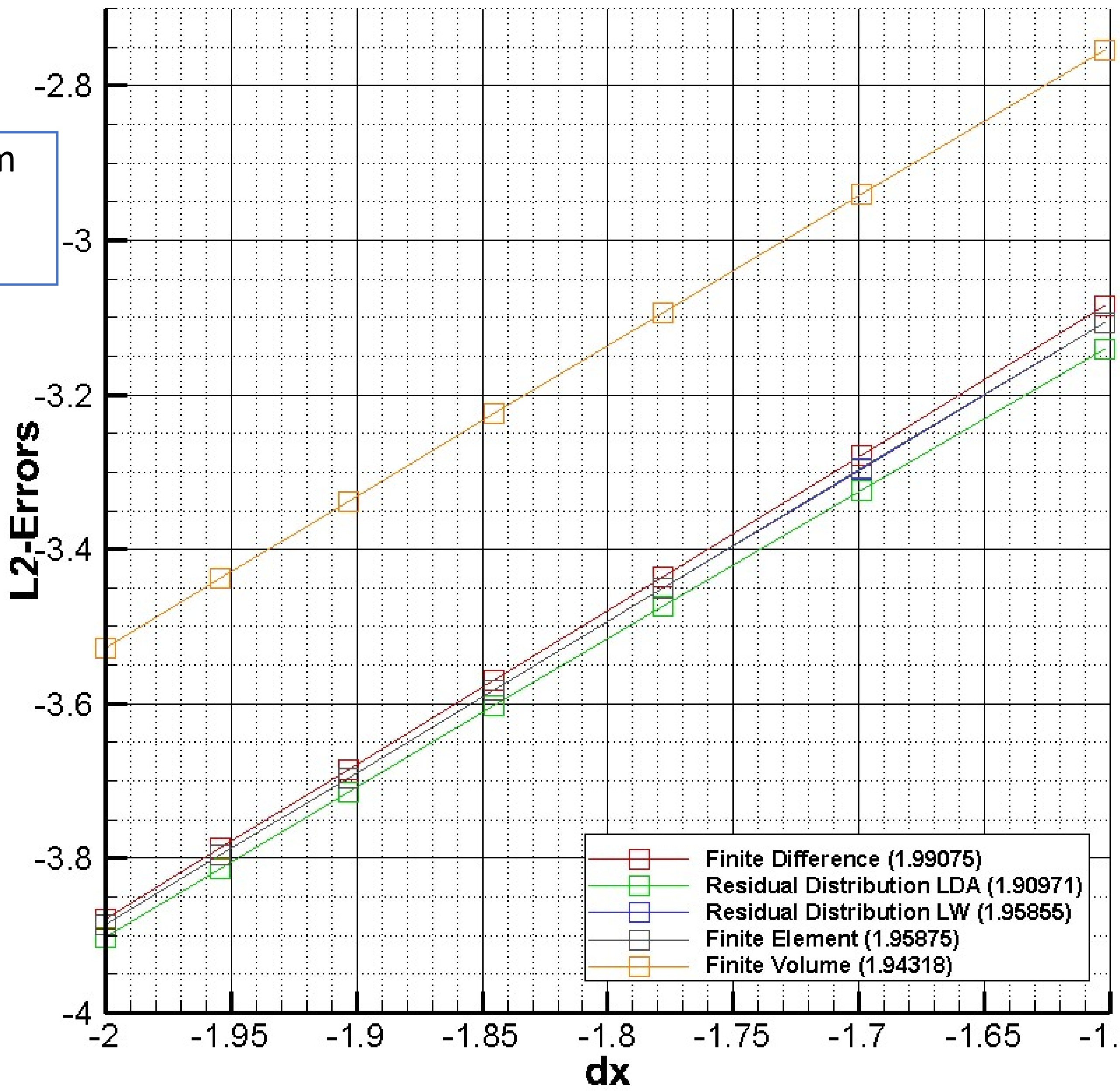


Vector Plot for (H_y, H_z)



L2-Errors

Grids vary from
40×40
to 100×100



Drawback

- The pseudo-time iteration could not work unless the initial guesses for each node are chosen properly (using Lax-Wendroff FD method).

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[1] C. Farhat and W. F. Durand (2015). *The Finite Volume Method* [Leture Notes]. Numerical Methods for Compressible Flows (AA214B), Farhat Research Group, Stanford University. Retrieved March 8, 2017 from https://web.stanford.edu/group/frg/course_work/AA214B.html

[2] H. Nishikawa (2008). *Multidimensional Reconstruction Schemes of Finite-Volume, Finite-Element, and Residual-Distribution*. CFD Notes – Computational Fluid Dynamics. Retrieved August 26, 2015 from http://ossanworld.com/cfdnotes/cfdnotes_rd_fv_fem.pdf

[3] N. Z. Mebrate (2007). *High Order Fluctuation Splitting Schemes for Hyperbolic Conservation Laws*. PhD Thesis for School of Computing, University of Leeds.

[4] J. D. Hoffman (2001). *Numerical Methods for Engineers and Scientists*. Second Edition (Revised and Expanded), Marcel Dekker Inc.

Appendix

- Finite Element Formulation
- Residual Distribution : Eigenvalues, Eigenvectors & Inflow Matrices

Finite Element Formulation

Governing Equations:

$$\frac{\partial H_z}{\partial x} + \varepsilon \frac{\partial E_y}{\partial y} = 0$$

$$\frac{\partial E_y}{\partial x} + \mu \frac{\partial H_z}{\partial y} = 0$$

i) Multiplying the Equations with weight function,
ii) Integrate over the control volume,
iii) Weak formulation,
iv) Conserved variables as interpolating basis function,
v) Galerkin's approach : weight function same as the basis function

i Weight function, w $= w(x, y)$

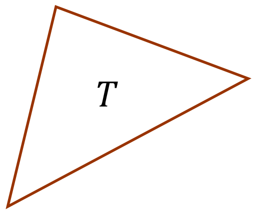
$$w \frac{\partial H_z}{\partial x} + \varepsilon w \frac{\partial E_y}{\partial y} = 0$$

$$w \frac{\partial E_y}{\partial x} + \mu w \frac{\partial H_z}{\partial y} = 0$$

ii Integration over the control volume

$$\iint_T \left(w \frac{\partial H_z}{\partial x} + \varepsilon w \frac{\partial E_y}{\partial y} \right) dxdy = 0$$

$$\iint_T \left(w \frac{\partial E_y}{\partial x} + \mu w \frac{\partial H_z}{\partial y} \right) dxdy = 0$$



iii Weak formulation (integration by parts from step (ii))

$$\iint_T \left(w \frac{\partial H_z}{\partial x} + \varepsilon w \frac{\partial E_y}{\partial y} \right) dxdy = \oint_{\partial T} \left(wH_z n_x + \varepsilon w E_y n_y \right) dl - \iint_T \left(H_z \frac{\partial w}{\partial x} + \varepsilon E_y \frac{\partial w}{\partial y} \right) dxdy$$

$$\iint_T \left(w \frac{\partial E_y}{\partial x} + \mu w \frac{\partial H_z}{\partial y} \right) dxdy = \oint_{\partial T} \left(wE_y n_x + \mu w H_z n_y \right) dl - \iint_T \left(E_y \frac{\partial w}{\partial x} + \mu H_z \frac{\partial w}{\partial y} \right) dxdy$$

$$\oint_{\partial T} w\left(H_z\, dy - \varepsilon E_y\, dx\right) - \int_T \left(H_z \frac{\partial w}{\partial x} + \varepsilon E_y \frac{\partial w}{\partial y}\right) dxdy = 0$$

$$\oint_{\partial T} w\left(E_y dy - \mu H_z dx\right) - \int_T \left(E_y \frac{\partial w}{\partial x} + \mu H_z \frac{\partial w}{\partial y}\right) dxdy = 0$$

The test function is equal to unity at node i but zero otherwise. Therefore, $\mathrm{w} = 0$ along the boundary.

iv Conserved variables in terms of basis interpolating function

$$E_{iy}(x,y) = E_{iy}\psi_i(x,y)$$

$$H_{iz}(x,y) = H_{iz}\psi_i(x,y)$$

$$H_{iz}\int_T \psi_i \frac{\partial w}{\partial x} dxdy + E_{iy}\int_T \varepsilon\psi_i \frac{\partial w}{\partial y} dxdy = 0$$

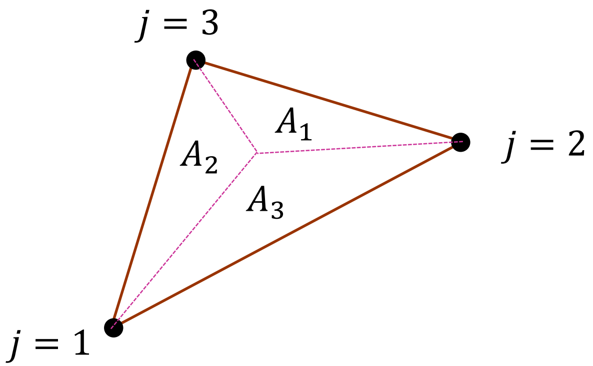
$$E_{iy}\int_T \psi_i \frac{\partial w}{\partial x} dxdy + H_{iz}\int_T \mu\psi_i \frac{\partial w}{\partial y} dxdy = 0$$

v $w(x,y) = \psi_j(x,y)$

$$H_{iz}\int_T \psi_i \frac{\partial \psi_j}{\partial x} dxdy + E_{iy}\int_T \varepsilon\psi_i \frac{\partial \psi_j}{\partial y} dxdy = 0$$

$$E_{iy}\int_T \psi_i \frac{\partial \psi_j}{\partial x} dxdy + H_{iz}\int_T \mu\psi_i \frac{\partial \psi_j}{\partial y} dxdy = 0$$

Basis functions:



Area coordinate, $L_j^e(x,y)$

$$\psi_j^e(x,y) = \frac{L_j^e(x,y)}{A}$$

$$L_1^e(x,y) = \frac{1}{2}\hat{z}\cdot(\vec{r}_3^{\,e} - \vec{r}_2^{\,e}) \times (\vec{r} - \vec{r}_2^{\,e})$$

$$L_2^e(x,y) = \frac{1}{2}\hat{z}\cdot(\vec{r}_1^{\,e} - \vec{r}_3^{\,e}) \times (\vec{r} - \vec{r}_3^{\,e})$$

$$L_3^e(x,y) = \frac{1}{2}\hat{z}\cdot(\vec{r}_2^{\,e} - \vec{r}_1^{\,e}) \times (\vec{r} - \vec{r}_1^{\,e})$$

or

$$L_j^e(x,y) = \frac{1}{2}\left(\vec{r} - \vec{r}_{j+1}^{\,e}\right)\cdot\hat{z}\times\vec{s}_i$$

$\vec{s}_j$ is the edge in the counter-clockwise direction opposing node j

$$\vec{s}_j = \vec{r}_{j-1}^{\,e} - \vec{r}_{j+1}^{\,e}$$

Gradients of the Local Basis Functions:

$$\nabla\psi_j^e(x,y) = \frac{\hat{z}\times\vec{s}_j}{2A}$$

and it is <u>constant</u> within each element.

Total Area of an Element:

$$A = \frac{1}{2}\hat{z}\cdot\vec{s}_2\times\vec{s}_3$$

Stiffness Matrix:

$$H_{iz}\int_T \psi_i \frac{\partial \psi_j}{\partial x}dxdy + E_{iy}\int_T \varepsilon\psi_i \frac{\partial \psi_j}{\partial y}dxdy = 0$$

$$E_{iy}\int_T \psi_i \frac{\partial \psi_j}{\partial x}dxdy + H_{iz}\int_T \mu\psi_i \frac{\partial \psi_j}{\partial y}dxdy = 0$$

The gradients of the local basis functions are always constant within each element.

$$H_{iz}\frac{\partial \psi_j}{\partial x}\int_T \psi_i dxdy + \varepsilon E_{iy}\frac{\partial \psi_j}{\partial y}\int_T \psi_i dxdy = 0$$

$$E_{iy}\frac{\partial \psi_j}{\partial x}\int_T \psi_i dxdy + \mu H_{iz}\frac{\partial \psi_j}{\partial y}\int_T \psi_i dxdy = 0$$

Evaluating Gradients of Local Basis Functions: $\frac{\partial \psi_i}{\partial x}$ $\frac{\partial \psi_i}{\partial y}$

Method 1: Using the definition

$$\nabla\psi_j^e(x,y) = \frac{\hat{z}\times\vec{s}_j}{2A}$$

$j = 1$:

$$\nabla\psi_1 = \frac{1}{2A}\hat{z}\times\vec{s}_1$$

$$= \frac{1}{2A}[-(y_3 - y_2)\hat{x} + (x_3 - x_2)\hat{y}]$$

$$\hat{z}\times\vec{s}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ (x_3 - x_2) & (y_3 - y_2) & 0 \end{vmatrix}$$

$$j = 2: \quad \nabla\psi_2 = \frac{1}{2A}\hat{z} \times \vec{s}_2$$

$$= \frac{1}{2A}[-(y_1 - y_3)\hat{x} + (x_1 - x_3)\hat{y}]$$

$$\hat{z} \times \vec{s}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ (x_1 - x_3) & (y_1 - y_3) & 0 \end{vmatrix}$$

$$j = 3: \quad \nabla\psi_3 = \frac{1}{2A}\hat{z} \times \vec{s}_3$$

$$= \frac{1}{2A}[-(y_2 - y_1)\hat{x} + (x_2 - x_1)\hat{y}]$$

$$\hat{z} \times \vec{s}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ (x_2 - x_1) & (y_2 - y_1) & 0 \end{vmatrix}$$

$$\because \nabla\psi_j = \frac{\partial\psi_j}{\partial x}\hat{x} + \frac{\partial\psi_j}{\partial y}\hat{y}$$

j	$\frac{\partial\psi_j}{\partial x}$	$\frac{\partial\psi_j}{\partial y}$
1	$\frac{y_2 - y_3}{2A}$	$\frac{x_3 - x_2}{2A}$
2	$\frac{y_3 - y_1}{2A}$	$\frac{x_1 - x_3}{2A}$
3	$\frac{y_1 - y_2}{2A}$	$\frac{x_2 - x_1}{2A}$

Evaluating Gradients of Local Basis Functions: $\dfrac{\partial \psi_i}{\partial x}$ $\dfrac{\partial \psi_i}{\partial y}$

Method 2: Coordinate transformation

$$\psi_1 = \xi \qquad \psi_2 = \eta \qquad \psi_3 = 1 - \xi - \eta$$

(a) Express (x, y)-coordinate in terms of basis coordinate (ψ_1, ψ_2, ψ_3)

$$\psi_3 = 1 - \psi_1 - \psi_2$$

$$x = \sum_{j=1}^{3} x_j \psi_j(\xi, \eta) = x_1\psi_1 + x_2\psi_2 + x_3\psi_3 = (x_1 - x_3)\psi_1 + (x_2 - x_3)\psi_2 + x_3$$

$$y = \sum_{j=1}^{3} y_j \psi_j(\xi, \eta) = y_1\psi_1 + y_2\psi_2 + y_3\psi_3 = (y_1 - y_3)\psi_1 + (y_2 - y_3)\psi_2 + y_3$$

1

(b) Express $\left(\dfrac{\partial \psi_j^e}{\partial x}, \dfrac{\partial \psi_j^e}{\partial y}\right)$ in terms of $\left(\dfrac{\partial \psi_j^e}{\partial \xi}, \dfrac{\partial \psi_j^e}{\partial \eta}\right)$ using chain rule

$$\frac{\partial \psi_j^e}{\partial \xi} = \frac{\partial \psi_j^e}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \psi_j^e}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \psi_j^e}{\partial \eta} = \frac{\partial \psi_j^e}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \psi_j^e}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} \dfrac{\partial \psi_j^e}{\partial \xi} \\ \dfrac{\partial \psi_j^e}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \psi_j^e}{\partial x} \\ \dfrac{\partial \psi_j^e}{\partial y} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \dfrac{\partial \psi_j^e}{\partial x} \\ \dfrac{\partial \psi_j^e}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \dfrac{\partial \psi_j^e}{\partial \xi} \\ \dfrac{\partial \psi_j^e}{\partial \eta} \end{Bmatrix}$$

2

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix} = \begin{bmatrix} (x_1 - x_3) & (y_1 - y_3) \\ (x_2 - x_3) & (y_2 - y_3) \end{bmatrix}$$

$$J^{-1} = \frac{1}{|J|} \begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) \\ (x_3 - x_2) & (x_1 - x_3) \end{bmatrix} \qquad |J| = (x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)$$

(c) Finding the derivatives $\left(\frac{\partial \psi_j^e}{\partial \xi}, \frac{\partial \psi_j^e}{\partial \eta}\right)$ so that equation (2) is computable

$$\psi_1 = L_1 \qquad \psi_2 = L_2 \qquad \psi_3 = 1 - L_1 - L_2$$

$$\frac{\partial \psi_1^e}{\partial \xi} = \frac{\partial \psi_1^e}{\partial L_1} = 1 \qquad \frac{\partial \psi_1^e}{\partial \eta} = \frac{\partial \psi_1^e}{\partial L_2} = 0$$

$$\frac{\partial \psi_2^e}{\partial \xi} = \frac{\partial \psi_2^e}{\partial L_1} = 0 \qquad \frac{\partial \psi_2^e}{\partial \eta} = \frac{\partial \psi_2^e}{\partial L_2} = 1$$

$$\frac{\partial \psi_3^e}{\partial \xi} = \frac{\partial \psi_3^e}{\partial L_1} = -1 \qquad \frac{\partial \psi_3^e}{\partial \eta} = \frac{\partial \psi_3^e}{\partial L_2} = -1$$

$$\vec{s}_1 \times \vec{s}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ (x_3 - x_2) & (y_3 - y_2) & 0 \\ (x_1 - x_3) & (y_1 - y_3) & 0 \end{vmatrix} = [(x_3 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_3 - y_2)]\hat{z} \quad \Rightarrow \quad |J| = \hat{z} \cdot \vec{s}_1 \times \vec{s}_2 = 2A$$

$$\begin{Bmatrix} \frac{\partial \psi_1^e}{\partial x} \\ \frac{\partial \psi_1^e}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial \psi_1^e}{\partial \xi} \\ \frac{\partial \psi_1^e}{\partial \eta} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) \\ (x_3 - x_2) & (x_1 - x_3) \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} (y_2 - y_3) \\ (x_3 - x_2) \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial \psi_2^e}{\partial x} \\ \frac{\partial \psi_2^e}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial \psi_2^e}{\partial \xi} \\ \frac{\partial \psi_2^e}{\partial \eta} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) \\ (x_3 - x_2) & (x_1 - x_3) \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} (y_3 - y_1) \\ (x_1 - x_3) \end{Bmatrix} \quad 3$$

$$\begin{Bmatrix} \frac{\partial \psi_3^e}{\partial x} \\ \frac{\partial \psi_3^e}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial \psi_3^e}{\partial \xi} \\ \frac{\partial \psi_3^e}{\partial \eta} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) \\ (x_3 - x_2) & (x_1 - x_3) \end{bmatrix} \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} (y_1 - y_2) \\ (x_2 - x_1) \end{Bmatrix}$$

Evaluating Integral of Basis Functions: $\int_T \psi_j dxdy$

Method 1: Direct integration

$$\iint_T \psi_j dxdy = 2A \int_{L_2=0}^{1} \int_{L_1=0}^{1-L_2} L_j \, dL_1 dL_2$$

$\psi_1 = L_1$

$\psi_2 = L_2$

$\psi_3 = 1 - L_1 - L_2$

j	ψ_j	$2A \iint_T L_j \, dL_1 dL_2$
1	L_1	$2A \int_{L_2=0}^{1} \left[\frac{(L_1)^2}{2}\right]_{L_1=0}^{1-L_2} dL_2 = 2A \int_{L_2=0}^{1} \frac{(1-L_2)^2}{2} dL_2 = \frac{2A}{2}\left[\frac{(L_2)^3}{3} - \frac{2(L_2)^2}{2} + L_2\right]_0^1 = \frac{2A}{6}$
2	L_2	$2A \int_{L_2=0}^{1} L_2(1-L_2) \, dL_2 = 2A \int_{L_2=0}^{1} L_2 - (L_2)^2 \, dL_2 = 2A \left[\frac{(L_2)^2}{2} - \frac{(L_2)^3}{3}\right]_0^1 = \frac{2A}{6}$
3	L_3	$2A \int_{L_2=0}^{1} \left[L_1 - \frac{(L_1)^2}{2} - L_1 L_2\right]_{L_1=0}^{1-L_2} dL_2 = 2A \int_{L_2=0}^{1} \left[(1-L_2) - \frac{(1-L_2)^2}{2} - (1-L_2)L_2\right] dL_2 = \frac{2A}{6}$

Method 2: Exact integration formula

$$\iint_T \psi_j dxdy = \iint_T L_1^m L_2^n L_3^p \, dxdy = \frac{m!\,n!\,p!}{(m+n+p+2)!} 2A$$

j	ψ_j	$\iint_T L_1^m L_2^n L_3^p \, dxdy$
1	L_1	$\frac{1!\,0!\,0!}{(1+0+0+2)!} 2A = \frac{2A}{6}$
2	L_2	$\frac{0!\,1!\,0!}{(0+1+0+2)!} 2A = \frac{2A}{6}$
3	L_3	$\frac{0!\,0!\,1!}{(0+0+1+2)!} 2A = \frac{2A}{6}$

Eigenvalues, Eigenvectors & Inflow Matrices for Residual Distribution

Governing Equations:

$$\frac{\partial H_z}{\partial x} + \varepsilon \frac{\partial E_y}{\partial y} = 0$$

$$\frac{\partial E_y}{\partial x} + \mu \frac{\partial H_z}{\partial y} = 0$$

Conserved variables, $\vec{U} = \begin{pmatrix} E_y \\ H_z \end{pmatrix}$

Fluxes:

$$\vec{F}(\vec{U}) = \begin{pmatrix} H_z \\ E_y \end{pmatrix} \qquad \vec{G}(\vec{U}) = \begin{pmatrix} \varepsilon E_y \\ \mu H_z \end{pmatrix}$$

Jacobian of Fluxes:

$$\frac{\partial \vec{F}(\vec{U})}{\partial \vec{U}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \frac{\partial \vec{G}(\vec{U})}{\partial \vec{U}} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$$

Hyperbolic System of Equations (with pseudo-time):

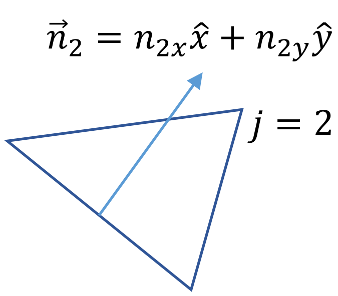
$$\frac{\partial \vec{U}}{\partial \tau} + \frac{\partial \vec{F}(\vec{U})}{\partial x} + \frac{\partial \vec{G}(\vec{U})}{\partial y} = 0$$

Inflow Matrix

$$K_j = \frac{1}{2}\left(\hat{x}\frac{\partial \vec{F}(\vec{U})}{\partial \vec{U}} + \hat{y}\frac{\partial \vec{G}(\vec{U})}{\partial \vec{U}}\right) \cdot \vec{n}_j$$

$$= \frac{1}{2}\left(n_{jx}\frac{\partial \vec{F}(\vec{U})}{\partial \vec{U}} + n_{jy}\frac{\partial \vec{G}(\vec{U})}{\partial \vec{U}}\right)$$

$$= \frac{1}{2}\begin{pmatrix} \varepsilon n_{jy} & n_{jx} \\ n_{jx} & \mu n_{jy} \end{pmatrix}$$



Eigenvalues

Let $A = \begin{pmatrix} \varepsilon n_{jy} & n_{jx} \\ n_{jx} & \mu n_{jy} \end{pmatrix}$

The characteristic polynomial could be obtained by letting the determinant of $A - \lambda I$ equal to zero.

$$\det(A - \lambda I) = \begin{vmatrix} \varepsilon n_{jy} - \lambda & n_{jx} \\ n_{jx} & \mu n_{jy} - \lambda \end{vmatrix} = 0$$

$$(\varepsilon n_{jy} - \lambda)(\mu n_{jy} - \lambda) - {n_{jx}}^2 = 0$$

$$\varepsilon \mu {n_{jy}}^2 - \lambda \mu n_{jy} - \lambda \varepsilon n_{jy} + \lambda^2 - {n_{jx}}^2 = 0$$

$$\lambda^2 + (-\mu n_{jy} - \varepsilon n_{jy})\lambda + (\varepsilon \mu {n_{jy}}^2 - {n_{jx}}^2) = 0$$

$$\lambda = \frac{(\mu n_{jy} + \varepsilon n_{jy})}{2} \pm \frac{\sqrt{(\mu n_{jy} + \varepsilon n_{jy})^2 - 4(\varepsilon \mu {n_{jy}}^2 - {n_{jx}}^2)}}{2}$$

Lemma : The eigenvalues λ is always real, $\lambda \in \mathbb{R}$, such that the system of equations is hyperbolic.

Proof :

$$(\mu n_{jy} + \varepsilon n_{jy})^2 - 4(\varepsilon \mu {n_{jy}}^2 - {n_{jx}}^2)$$

$$= \mu^2 {n_{jy}}^2 + \varepsilon^2 {n_{jy}}^2 + 2\mu\varepsilon {n_{jy}}^2 - 4\mu\varepsilon {n_{jy}}^2 + 4{n_{jx}}^2$$

$$= \mu^2 {n_{jy}}^2 + \varepsilon^2 {n_{jy}}^2 - 2\mu\varepsilon {n_{jy}}^2 + 4{n_{jx}}^2$$

$$= (\mu n_{jy} - \varepsilon n_{jy})^2 + 4{n_{jx}}^2 \quad > 0$$

Eigenvectors

$$\lambda_1 = \frac{(\mu n_{jy} + \varepsilon n_{jy})}{2} - \delta \qquad \text{where} \qquad \delta = \frac{\sqrt{(\mu n_{jy} - \varepsilon n_{jy})^2 + 4 n_{jx}{}^2}}{2}$$

$$(A - \lambda_1 I)\vec{x}_1 = \vec{0}$$

$$\begin{pmatrix} \varepsilon n_{jy} - \lambda_1 & n_{jx} \\ n_{jx} & \mu n_{jy} - \lambda_1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (\varepsilon n_{jy} - \lambda_1) x_{11} + n_{jx} x_{12} = 0 & (1) \\ n_{jx} x_{11} + (\mu n_{jy} - \lambda_1) x_{12} = 0 & (2) \end{cases}$$

Equations (1) and (2) are to be fulfilled simultaneously if x_{11} and x_{12} are chosen as

$$x_{11} = (\mu n_{jy} - \lambda_1) n_{jx}$$

$$x_{12} = -(\mu n_{jy} - \lambda_1)(\varepsilon n_{jy} - \lambda_1)$$

where we have made use of the substitution that

$$(\mu n_{jy} - \lambda_1)(\varepsilon n_{jy} - \lambda_1) = n_{jx}{}^2$$

Since x_{11} and x_{12} both contain $(\mu n_{jy} - \lambda_1)$, thus they can be simplified as

$$\vec{x}_1 = \begin{pmatrix} n_{jx} \\ \lambda_1 - \varepsilon n_{jy} \end{pmatrix}$$

The eigenvectors for λ_2 can be obtained using a similar way as

$$\lambda_2 = \frac{(\mu n_{jy} + \varepsilon n_{jy})}{2} + \delta$$

$$(A - \lambda_2 I)\vec{x}_2 = \vec{0}$$

$$\vec{x}_2 = \begin{pmatrix} n_{jx} \\ \lambda_2 - \varepsilon n_{jy} \end{pmatrix}$$

Right-Eigenvectors

The right-eigenvector is then given as

$$R = (\vec{x}_1, \vec{x}_2) = \begin{pmatrix} n_{jx} & n_{jx} \\ \lambda_1 - \varepsilon n_{jy} & \lambda_2 - \varepsilon n_{jy} \end{pmatrix}$$

Left-Eigenvectors

The left-eigenvector is just the inverse of R, meaning that $L = R^{-1}$

$$\det(R) = (\lambda_2 n_{jx} - \varepsilon n_{jx} n_{jy}) - (\lambda_1 n_{jx} - \varepsilon n_{jx} n_{jy})$$

$$= -n_{jx}\sqrt{(\mu n_{jy} - \varepsilon n_{jy})^2 + 4{n_{jx}}^2}$$

$$= -2\delta n_{jx}$$

$$L = R^{-1}$$

$$= -2\delta n_{jx} \begin{pmatrix} \lambda_2 - \varepsilon n_{jy} & -n_{jx} \\ \varepsilon n_{jy} - \lambda_1 & n_{jx} \end{pmatrix}$$