Dimensionless 2D Steady Maxwell's Equation Using Jacobi's Iteration

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- **Dimensionless 2D Maxwell's Equations**
- **The Numerical Solvers in Matrix Form**
  a) Residual Distribution
  b) Finite Volume
  c) Finite Element
- **Numerical Results**

Motivation

Problems from previous presentation:

- Inconsistency in dimension.
- Adding the pseudo-time to the equations might pose some issues.
- Galerkin's finite-element method seems to work.

Dimensionless Equations

- Scaling of the 2D steady Maxwell's equations to its dimensionless form
- The scaling version of analytical solution

1D Maxwell's Equation

The Maxwell's equation in 1D is given as

$$\frac{\partial H_{zs}}{\partial x_s} + \varepsilon \frac{\partial E_{ys}}{\partial t_s} = 0 \qquad \text{1a}$$

$$\frac{\partial E_{ys}}{\partial x_s} + \mu \frac{\partial H_{zs}}{\partial t_s} = 0 \qquad \text{1b}$$

The subscript s denotes that those variables are having their corresponding physical units.

$[x_s] = m$ (metre)

$[t_s] = s$ (second)

$[H_{zs}] = A\,m^{-1}$ (Ampere per metre)

$[E_{ys}] = V\,m^{-1}$ (Volt per metre)

Introducing the scaled independent variables,

$$x = \frac{x_s}{L} \qquad \text{2a}$$

$$t = \frac{c}{L} t_s \qquad \text{2b}$$

Secondly, the vector fields are scaled as following:

$$E_{ys}(x,t) = \widetilde{H}_0 Z E_y(x,t) \qquad \text{2c}$$

$$H_{zs}(x,t) = \widetilde{H}_0 H_z(x,t) \qquad \text{2d}$$

where $\widetilde{H}_0$ is the reference magnetic field strength (A/m).

c is the speed of light in the medium where it propagates and Z is the intrinsic impedance of the medium.

$$Z = \sqrt{\frac{\mu}{\varepsilon}} \qquad \text{3a}$$

$$c = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{3b}$$

Substituting all the scaled quantities in equations (2) into equations (1), the 1D Maxwell's equations become a set of dimensionless equations

$$\left(\frac{1}{L}\widetilde{H}_0\right)\frac{\partial H_z}{\partial x} + \left(\varepsilon\frac{c}{L}\widetilde{H}_0 Z\right)\frac{\partial E_y}{\partial t} = 0 \qquad \text{4a}$$

$$\left(\frac{1}{L}\widetilde{H}_0 Z\right)\frac{\partial E_y}{\partial x} + \left(\mu\frac{c}{L}\widetilde{H}_0\right)\frac{\partial H_z}{\partial t} = 0 \qquad \text{4b}$$

Equations (4) when undergo mapping transformation, $t \rightarrow y$ gives

$$\left(\frac{1}{L}\widetilde{H}_0\right)\frac{\partial H_z}{\partial x} + \left(\varepsilon\frac{c}{L}\widetilde{H}_0 Z\right)\frac{\partial E_y}{\partial y} = 0$$

$$\left(\frac{1}{L}\widetilde{H}_0 Z\right)\frac{\partial E_y}{\partial x} + \left(\mu\frac{c}{L}\widetilde{H}_0\right)\frac{\partial H_z}{\partial y} = 0$$

By letting $a = \left(\widetilde{H}_0/L\right)$ and $b = \left(\widetilde{H}_0 Z/L\right)$, then we will have the simplified form of equations

$$a\frac{\partial H_z}{\partial x} + a\frac{\partial E_y}{\partial y} = 0 \qquad \text{5a}$$

$$b\frac{\partial E_y}{\partial x} + b\frac{\partial H_z}{\partial y} = 0 \qquad \text{5b}$$

Remarks:

Variables **with** subscript s include their corresponding physical units : x_s, t_s, E_{ys}, H_{zs}

Variables **without** subscript s are dimensionless scaled variables: x, t, E_y, H_z

Analytical Solutions

Originally, the 1D Maxwell's equations have the analytical solution of

$$H_{zs(m)}(x,t) = \widetilde{H}_{0m} \cos \frac{m\pi x_s}{L} \cos \omega_m t_s$$ 6a

$$E_{ys(m)}(x,t) = \frac{\widetilde{H}_{0m}}{\varepsilon \omega_m} \frac{m\pi}{L} \sin \frac{m\pi x_s}{L} \sin \omega_m t_s$$ 6b

where the angular frequency is given as

$$\omega_m = \frac{mc\pi}{L}$$

mode of propagation

By introducing the scaled variables in equations (2) and also the coordinate transformation $t \rightarrow y$, the dimensionless Maxwell's equations have the analytical solution of

$$H_{z(m)}(x,t) = \cos(m\pi x)\cos(m\pi y)$$ 7a

$$E_{y(m)}(x,t) = \sin(m\pi x)\sin(m\pi y)$$ 7b

Residual Distribution

- Basic of the RD scheme (pseudo-time)
- RD scheme in stiffness matrix form
- Quick view of the Jacobi's iteration

Unsteady Equations

$$\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{\mathfrak{F}} = 0$$

$$S_i \frac{\partial \vec{U}}{\partial t} + \sum_{T \in \cup \Delta_i} \mathbb{B}_i^T \, \Phi^T = 0$$

Steady Equations

$$\nabla \cdot \vec{\mathfrak{F}} = 0$$

$$\sum_{T \in \cup \Delta_i} \mathbb{B}_i^T \, \Phi^T = 0$$

Solving technique : pseudo-time iteration

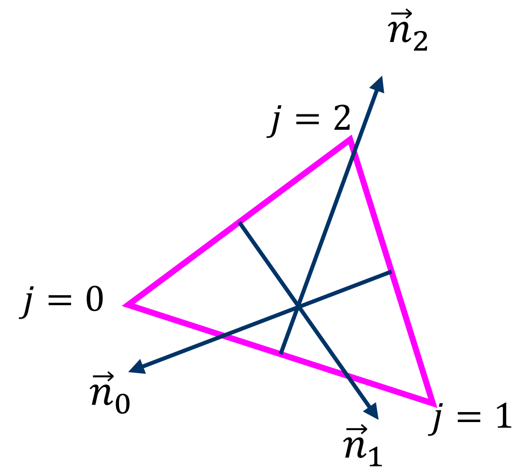
Proposed by Jameson in 1991 at handling FV solver.

$$S_i \frac{\partial \vec{U}}{\partial t} + \sum_{T \in \cup \Delta_i} \mathbb{B}_i^T \, \Phi^T = 0 \quad \Rightarrow \quad S_i \frac{\partial \vec{U}}{\partial \tau} + \sum_{T \in \cup \Delta_i} \mathbb{B}_i^T \, \Phi^T = 0$$

Pseudo-time Iterations

$$\vec{U}_i^{(k+1)} = \vec{U}_i^{(k)} - \frac{\Delta\tau}{S_i} \sum_{T \in \cup \Delta_i} \mathbb{B}_i^T \, \Phi^T$$

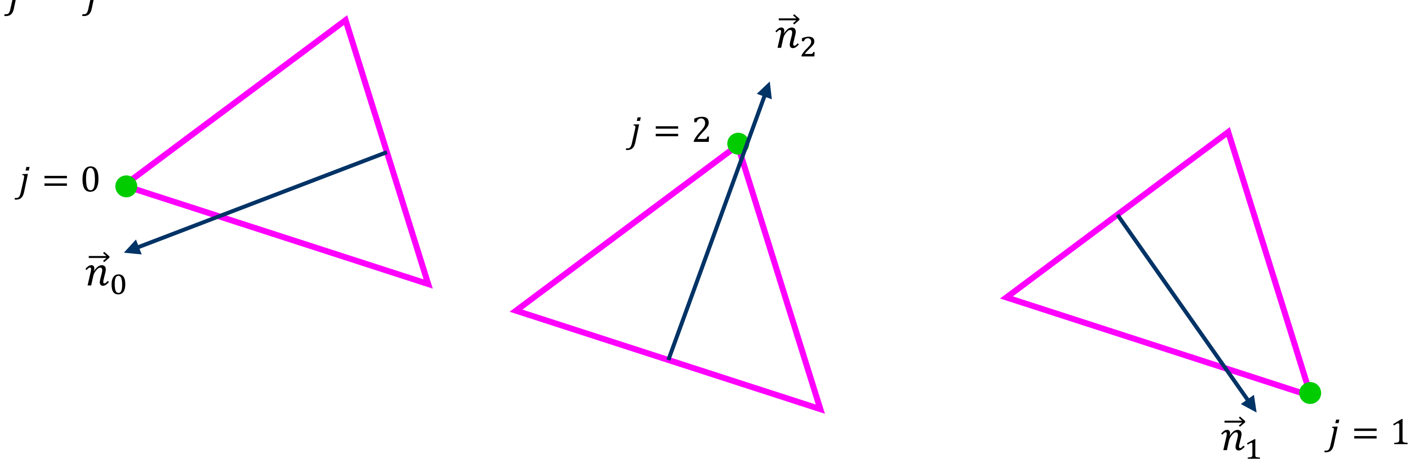
$$\vec{U}_i^{(k+1)} = \vec{U}_i^{(k)} - \frac{\Delta\tau}{S_i} \sum_{T \in \cup \Delta_i} \mathbb{B}_i^T \sum_{j \in T} \frac{1}{2} \vec{\mathfrak{F}}_j \cdot \vec{n}_j$$

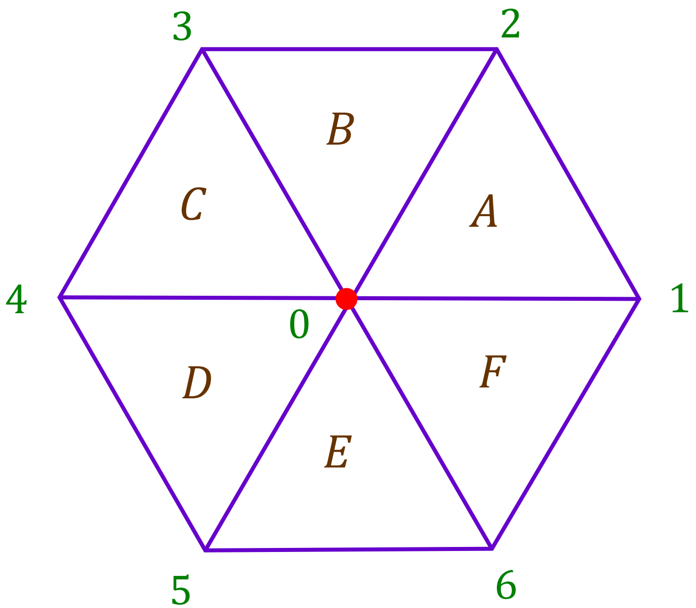


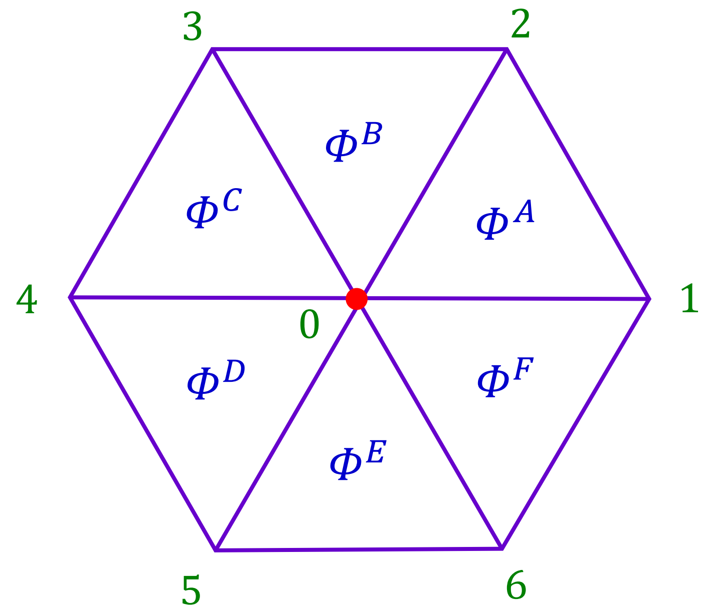
Scaled inward normals

Cell Residuals

$$\Phi^T = \frac{1}{2} \sum_{j \in T} \vec{\mathfrak{F}}_j \cdot \vec{n}_j$$





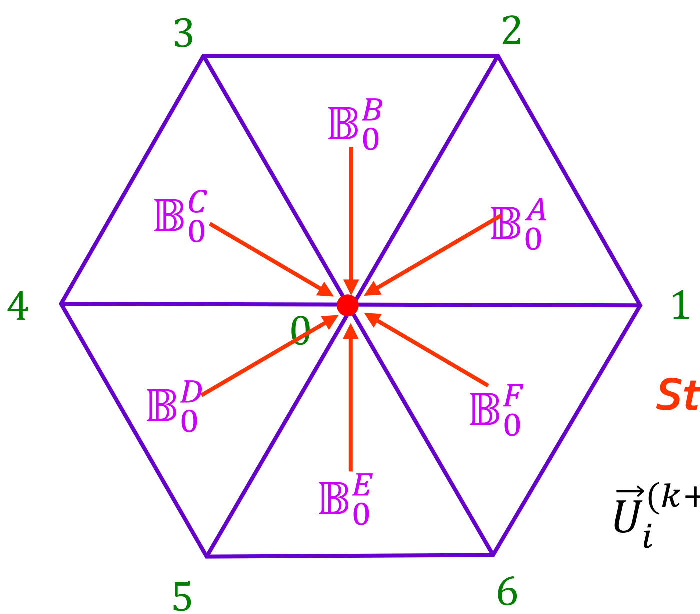


Labelling of global nodes around $i = 0$

Elements $T = A, B, C, D, E, F$

Step 1 : Calculate cell residual

$$\Phi^T = \sum_{j \in T} \frac{1}{2} \vec{\mathfrak{F}}_j \cdot \vec{n}_j$$



Step 2 : Distribute residual

$$\vec{U}_i^{(k+1)} = \vec{U}_i^{(k)} - \frac{\Delta\tau}{S_i} \sum_{T \in \cup\Delta_i} \mathbb{B}_i^T \, \Phi^T$$

Dimensionless 2D Steady Maxwell's Equations

$$a\frac{\partial H_z}{\partial x} + a\frac{\partial E_y}{\partial y} = 0$$

$$b\frac{\partial E_y}{\partial x} + b\frac{\partial H_z}{\partial y} = 0$$

Fluxes

$$\vec{\mathfrak{F}} = \hat{x}F(\vec{U}) + \hat{y}G(\vec{U})$$

$$\vec{\mathfrak{F}} = \hat{x}\begin{pmatrix} aH_z \\ bE_y \end{pmatrix} + \hat{y}\begin{pmatrix} aE_y \\ bH_z \end{pmatrix}$$

$$\vec{F}(\vec{U}) = \begin{pmatrix} aH_z \\ bE_y \end{pmatrix} \qquad \vec{G}(\vec{U}) = \begin{pmatrix} aE_y \\ bH_z \end{pmatrix}$$

Residual

$$\Phi^T = \frac{1}{2}\sum_{j\in T}\begin{pmatrix} an_{jx} H_{zj} + an_{jy}E_{yj} \\ bn_{jx}E_{yj} + bn_{jy}H_{zj} \end{pmatrix}$$

$$\nabla \cdot \vec{\mathfrak{F}} = 0$$ *Steady*

$$\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{\mathfrak{F}} = 0$$ *Unsteady*

$$\frac{\partial \vec{U}}{\partial \tau} + \nabla \cdot \vec{\mathfrak{F}} = 0$$ *Pseudo-time for Steady*

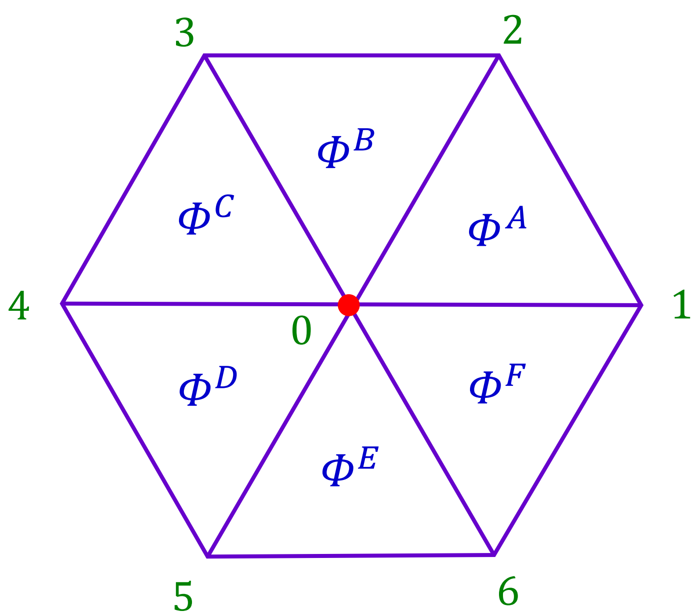
Solving using Jacobi's method :

solve the system of equations without introducing pseudo-time

$$\left.\begin{aligned} a\frac{\partial H_z}{\partial x} + a\frac{\partial E_y}{\partial y} = 0 \\ b\frac{\partial E_y}{\partial x} + b\frac{\partial H_z}{\partial y} = 0 \end{aligned}\right\} \sum_{T\in\cup\Delta_i} \mathbb{B}_i^T\,\Phi^T = 0$$

Conserved variables

Inflow matrix $\mathbb{K}_j^T = \frac{1}{2}\begin{pmatrix} an_{jy} & an_{jx} \\ bn_{jx} & bn_{jy} \end{pmatrix}$ $\qquad \vec{U}_j = \begin{pmatrix} E_{yj} \\ H_{zj} \end{pmatrix}$



$$\begin{aligned} \sum_{T\in\cup\Delta_i} \mathbb{B}_i^T\,\Phi^T = \{ & \mathbb{B}_0^A[\mathbb{K}_0^A\vec{U}_0 + \mathbb{K}_1^A\vec{U}_1 + \mathbb{K}_2^A\vec{U}_2] \\ & +\mathbb{B}_0^B[\mathbb{K}_0^B\vec{U}_0 + \mathbb{K}_2^B\vec{U}_2 + \mathbb{K}_3^B\vec{U}_3] \\ & +\mathbb{B}_0^C[\mathbb{K}_0^C\vec{U}_0 + \mathbb{K}_3^C\vec{U}_3 + \mathbb{K}_4^C\vec{U}_4] \\ & +\mathbb{B}_0^D[\mathbb{K}_0^D\vec{U}_0 + \mathbb{K}_4^D\vec{U}_4 + \mathbb{K}_5^D\vec{U}_5] \\ & +\mathbb{B}_0^E[\mathbb{K}_0^E\vec{U}_0 + \mathbb{K}_5^E\vec{U}_5 + \mathbb{K}_6^E\vec{U}_6] \\ & +\mathbb{B}_0^F[\mathbb{K}_0^F\vec{U}_0 + \mathbb{K}_6^F\vec{U}_6 + \mathbb{K}_1^F\vec{U}_1]\} = 0 \end{aligned}$$

$$
\begin{bmatrix}
M_0 & M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix} \vec{U}_0 \\ \vec{U}_1 \\ \vec{U}_2 \\ \vec{U}_3 \\ \vec{U}_4 \\ \vec{U}_5 \\ \vec{U}_6 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$M_0 = \mathbb{B}_0^A \mathbb{K}_0^A + \mathbb{B}_0^B \mathbb{K}_0^B + \mathbb{B}_0^C \mathbb{K}_0^C + \mathbb{B}_0^D \mathbb{K}_0^D + \mathbb{B}_0^E \mathbb{K}_0^E + \mathbb{B}_0^F \mathbb{K}_0^F$$

$$M_1 = \mathbb{B}_0^A \mathbb{K}_1^A + \mathbb{B}_0^F \mathbb{K}_1^F$$

$$M_2 = \mathbb{B}_0^A \mathbb{K}_2^A + \mathbb{B}_0^B \mathbb{K}_2^B$$

$$M_3 = \mathbb{B}_0^B \mathbb{K}_3^B + \mathbb{B}_0^C \mathbb{K}_3^C$$

$$M_4 = \mathbb{B}_0^C \mathbb{K}_4^C + \mathbb{B}_0^D \mathbb{K}_4^D$$

$$M_5 = \mathbb{B}_0^D \mathbb{K}_5^D + \mathbb{B}_0^E \mathbb{K}_5^E$$

$$M_6 = \mathbb{B}_0^E \mathbb{K}_6^E + \mathbb{B}_0^F \mathbb{K}_6^F$$

Jacobi's Method

- Jacobi's method is one of the simplest way to solve a matrix iteratively.
- It is very useful especially in handling sparse matrix.
- No inversion of matrix is involved.

To solve for i-th equation of linear system $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix}$$

Provided the diagonal term of the row $a_{ii} \neq 0$

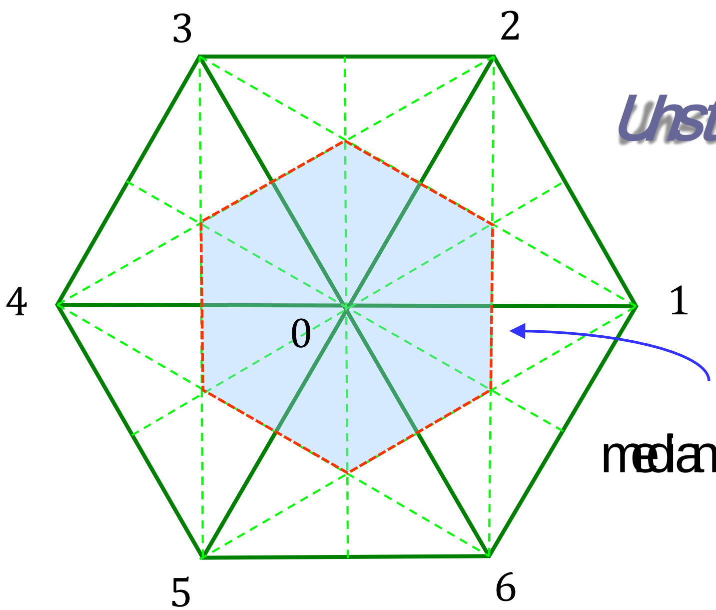
$$x_i = \frac{b_i}{a_{ii}} + \sum_{\substack{j=0 \\ j \neq i}}^{n} \left(-\frac{a_{ij} x_j}{a_{ii}} \right) \qquad \text{for} \qquad i = 0, 1, 2, \cdots, n$$

Iterates the linear system numerically in k-counter, gives:

$$x_i^{(k)} = \frac{b_i}{a_{ii}} + \sum_{\substack{j=0 \\ j \neq i}}^{n} \left(-\frac{a_{ij} x_j^{(k-1)}}{a_{ii}} \right) \qquad \text{for} \qquad i = 0, 1, 2, \cdots, n$$

Finite Volume

- Basic of the FV scheme (pseudo-time)
- FV scheme in stiffness matrix form



median dual cell

Unsteady

$$S_i \frac{\partial \vec{U}}{\partial t} + \sum_{j \in \cup k_i} \left[\vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^L\right) + \vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^R\right) \right] = 0$$

numerical flux

$\vec{n}_{ij}^L$

$$\vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^L\right) = \frac{\vec{\mathfrak{F}}_i \cdot \vec{n}_{ij}^L + \vec{\mathfrak{F}}_j \cdot \vec{n}_{ij}^L}{2} - \frac{1}{2} \left| A_{ij}^L \right| \left(\vec{U}_j - \vec{U}_i \right)$$

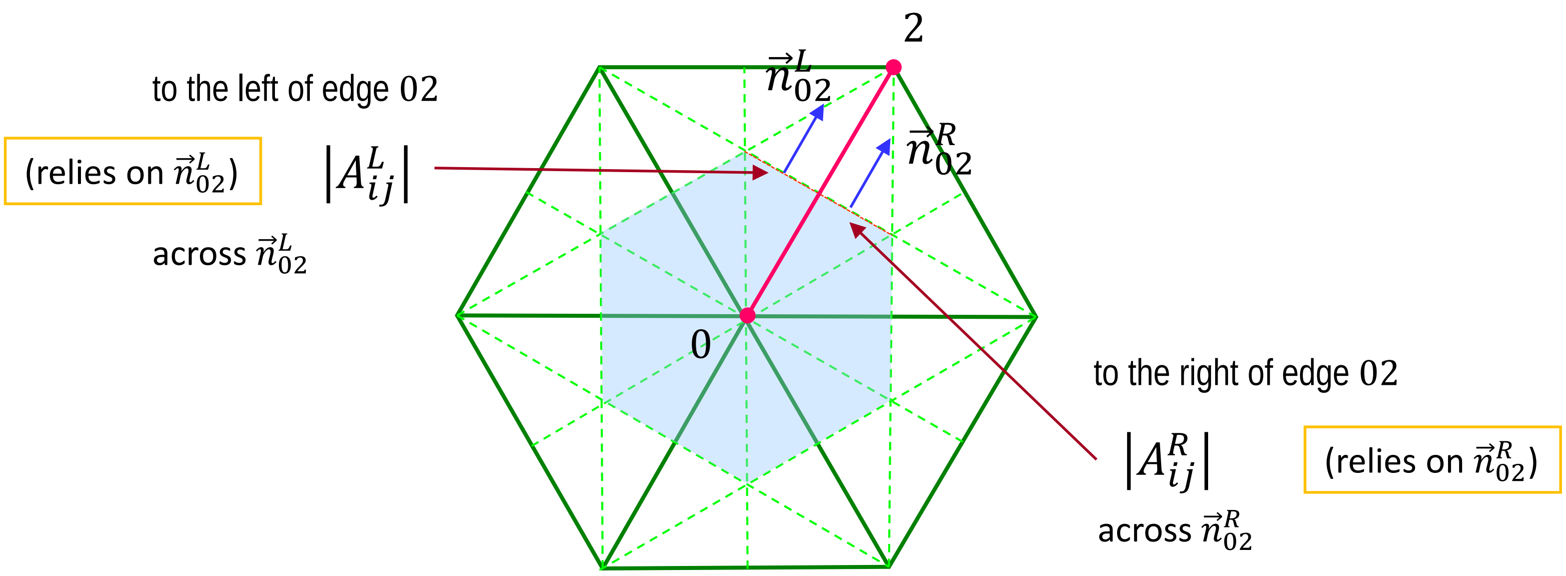
$\vec{n}_{ij}^R$

$$\vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^R\right) = \frac{\vec{\mathfrak{F}}_i \cdot \vec{n}_{ij}^R + \vec{\mathfrak{F}}_j \cdot \vec{n}_{ij}^R}{2} - \frac{1}{2} \left| A_{ij}^R \right| \left(\vec{U}_j - \vec{U}_i \right)$$

Numerical flux for Finite Volume

Absolute inflow matrix

$$|A| = R\,|\Lambda|\,L$$



Normal to the median dual cell

$$\vec{H}(\vec{U}_0, \vec{U}_2, \vec{n}^L_{02}) + \vec{H}(\vec{U}_0, \vec{U}_2, \vec{n}^R_{02}) = \frac{\vec{\mathfrak{F}}_0 \cdot \vec{n}^L_{02} + \vec{\mathfrak{F}}_2 \cdot \vec{n}^L_{02}}{2} - \frac{1}{2}|A^L_{02}|(\vec{U}_2 - \vec{U}_0)$$

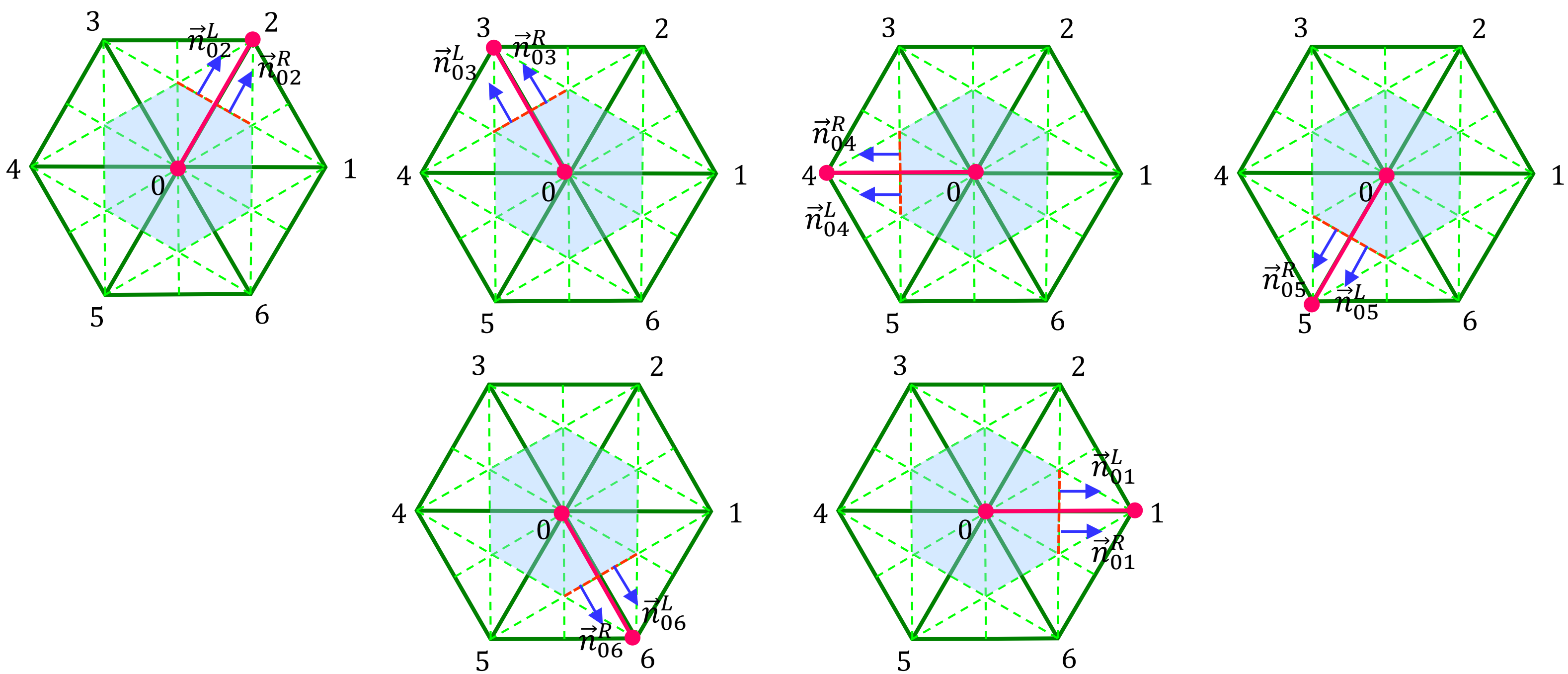
$$+ \frac{\vec{\mathfrak{F}}_0 \cdot \vec{n}^R_{02} + \vec{\mathfrak{F}}_2 \cdot \vec{n}^R_{02}}{2} - \frac{1}{2}|A^R_{02}|(\vec{U}_2 - \vec{U}_0)$$

pseudo-time iteration

(steady)

$$S_i \frac{\partial \vec{U}}{\partial \tau} + \sum_{j \in \cup k_i} \left[\vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^L\right) + \vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^R\right) \right] = 0$$

$$\vec{U}_i^{(k+1)} = \vec{U}_i^{(k)} - \frac{\Delta \tau}{S_i} \sum_{j \in \cup k_i} \left[\vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^L\right) + \vec{H}\left(\vec{U}_i, \vec{U}_j, \vec{n}_{ij}^R\right) \right] = 0$$



Jacobi's iteration
(steady)

$$\sum_{j\in\cup k_i}\left[\vec{H}\left(\vec{U}_i,\vec{U}_j,\vec{n}_{ij}^L\right)+\vec{H}\left(\vec{U}_i,\vec{U}_j,\vec{n}_{ij}^R\right)\right]=0$$

$$\sum_{j\in\cup k_i}\left[\vec{H}\left(\vec{U}_i,\vec{U}_j,\vec{n}_{ij}^L\right)+\vec{H}\left(\vec{U}_i,\vec{U}_j,\vec{n}_{ij}^R\right)\right]=$$

$$\frac{1}{2}\begin{bmatrix} a\left(n_{01,y}^L+n_{01,y}^R\right) & a\left(n_{01,x}^L+n_{01,x}^R\right) \\ b\left(n_{01,x}^L+n_{01,x}^R\right) & b\left(n_{01,y}^L+n_{01,y}^R\right)\end{bmatrix}\left(\boxed{\vec{U}_0}+\boxed{\vec{U}_1}\right)-\frac{1}{2}\left[\left|A_{01}^L\right|+\left|A_{01}^R\right|\right]\left(\boxed{\vec{U}_1}-\boxed{\vec{U}_0}\right)$$

$$+\frac{1}{2}\begin{bmatrix} a\left(n_{02,y}^L+n_{02,y}^R\right) & a\left(n_{02,x}^L+n_{02,x}^R\right) \\ b\left(n_{02,x}^L+n_{02,x}^R\right) & b\left(n_{02,y}^L+n_{02,y}^R\right)\end{bmatrix}\left(\boxed{\vec{U}_0}+\vec{U}_2\right)-\frac{1}{2}\left[\left|A_{02}^L\right|+\left|A_{02}^R\right|\right]\left(\vec{U}_2-\boxed{\vec{U}_0}\right)$$

$$+\frac{1}{2}\begin{bmatrix} a\left(n_{03,y}^L+n_{03,y}^R\right) & a\left(n_{03,x}^L+n_{03,x}^R\right) \\ b\left(n_{03,x}^L+n_{03,x}^R\right) & b\left(n_{03,y}^L+n_{03,y}^R\right)\end{bmatrix}\left(\boxed{\vec{U}_0}+\vec{U}_3\right)-\frac{1}{2}\left[\left|A_{03}^L\right|+\left|A_{03}^R\right|\right]\left(\vec{U}_3-\boxed{\vec{U}_0}\right)$$

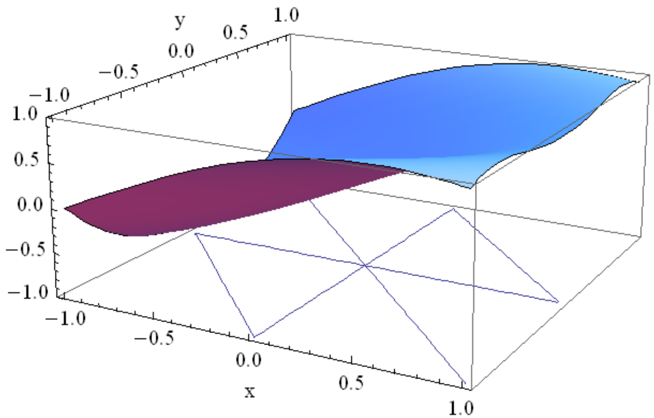
$$+\frac{1}{2}\begin{bmatrix} a\left(n_{04,y}^L+n_{04,y}^R\right) & a\left(n_{04,x}^L+n_{04,x}^R\right) \\ b\left(n_{04,x}^L+n_{04,x}^R\right) & b\left(n_{04,y}^L+n_{04,y}^R\right)\end{bmatrix}\left(\boxed{\vec{U}_0}+\vec{U}_4\right)-\frac{1}{2}\left[\left|A_{04}^L\right|+\left|A_{04}^R\right|\right]\left(\vec{U}_4-\boxed{\vec{U}_0}\right)$$

$$+\frac{1}{2}\begin{bmatrix} a\left(n_{05,y}^L+n_{05,y}^R\right) & a\left(n_{05,x}^L+n_{05,x}^R\right) \\ b\left(n_{05,x}^L+n_{05,x}^R\right) & b\left(n_{05,y}^L+n_{05,y}^R\right)\end{bmatrix}\left(\boxed{\vec{U}_0}+\vec{U}_5\right)-\frac{1}{2}\left[\left|A_{05}^L\right|+\left|A_{05}^R\right|\right]\left(\vec{U}_5-\boxed{\vec{U}_0}\right)$$

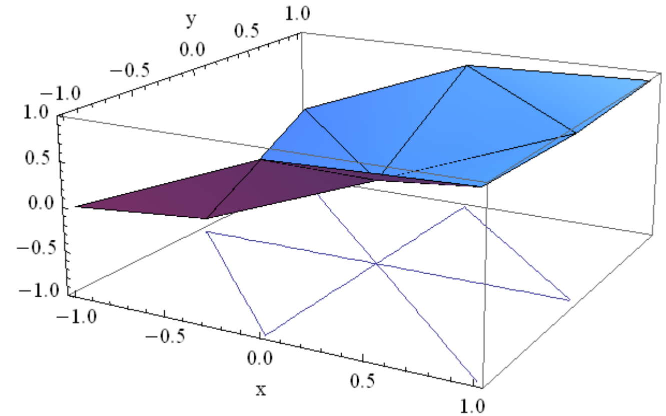
$$+\frac{1}{2}\begin{bmatrix} a\left(n_{06,y}^L+n_{06,y}^R\right) & a\left(n_{06,x}^L+n_{06,x}^R\right) \\ b\left(n_{06,x}^L+n_{06,x}^R\right) & b\left(n_{06,y}^L+n_{06,y}^R\right)\end{bmatrix}\left(\boxed{\vec{U}_0}+\vec{U}_6\right)-\frac{1}{2}\left[\left|A_{06}^L\right|+\left|A_{06}^R\right|\right]\left(\vec{U}_6-\boxed{\vec{U}_0}\right)=0$$

Finite Element

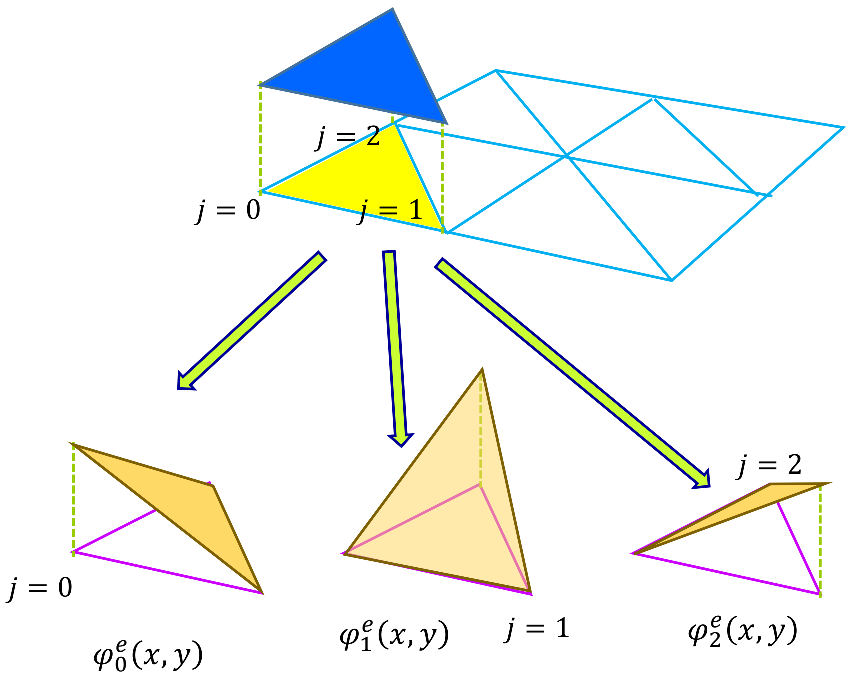
- Fundamentals of FE method
- Classification of FE method



Linear interpolation
for P1 element



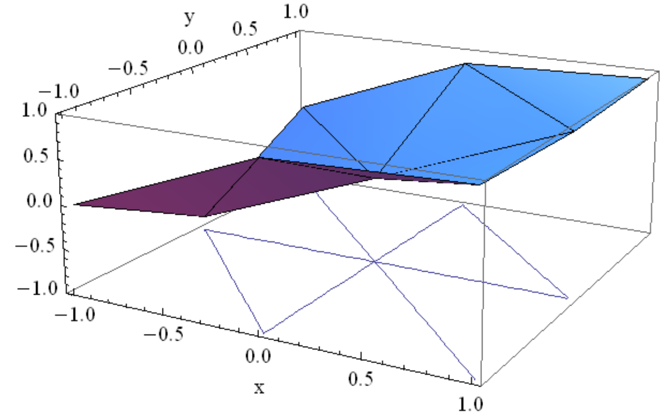
- approximate the function $u(x, y)$ over the triangle T as a linear plane.
- this linear plane is a combination of three interpolating functions.
- The basis functions have the following properties:
  i. They are linear in x and y inside each element
  $$\varphi_j^T(x, y) = a_j^T + b_j^T x + c_j^T y$$
  ii. They equal to unity on one node and vanish on the others
  $$\varphi_i^T(x_i, y_i) = 1$$
  $$\varphi_i^T(x_j, y_j) = 0 \quad , \qquad \forall\, i \neq j$$



$$a\frac{\partial H_z}{\partial x} + a\frac{\partial E_y}{\partial y} = 0$$

$$b\frac{\partial E_y}{\partial x} + b\frac{\partial H_z}{\partial y} = 0$$

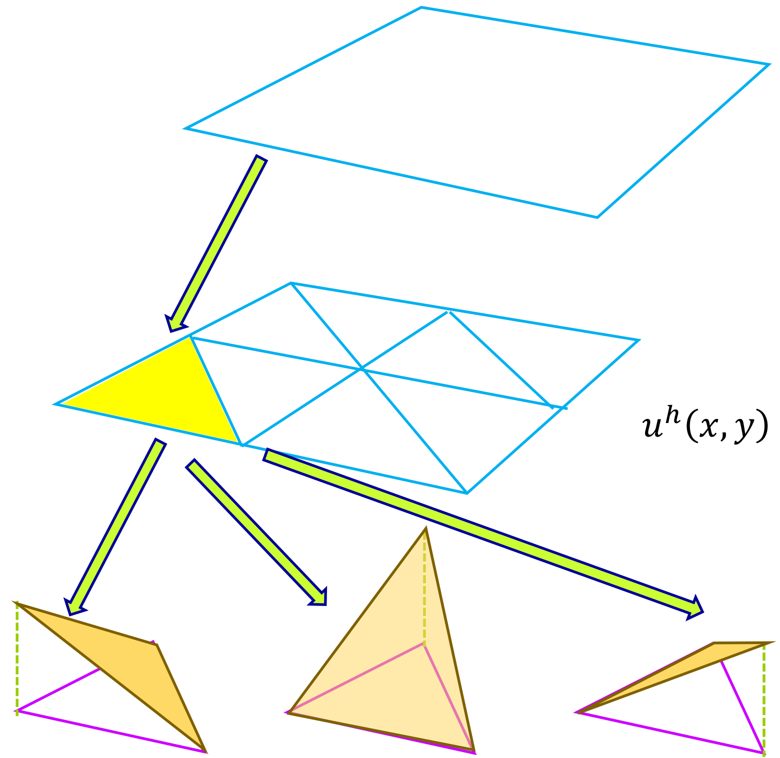
$$\nabla \cdot \vec{\mathfrak{F}} = 0$$



$$\nabla \cdot \vec{\mathfrak{F}}^h = 0$$

$$\sum_{T \in \cup \Delta_i} \nabla \cdot \vec{\mathfrak{F}}^T = 0$$

$$\sum_{T \in \cup \Delta_i} \sum_{j \in T} \nabla \cdot \vec{\mathfrak{F}}_j^T = 0$$



$$u^h(x, y)$$

$$u^h(x, y) \cong \sum_{T \in \Omega_h} u^T(x, y)$$

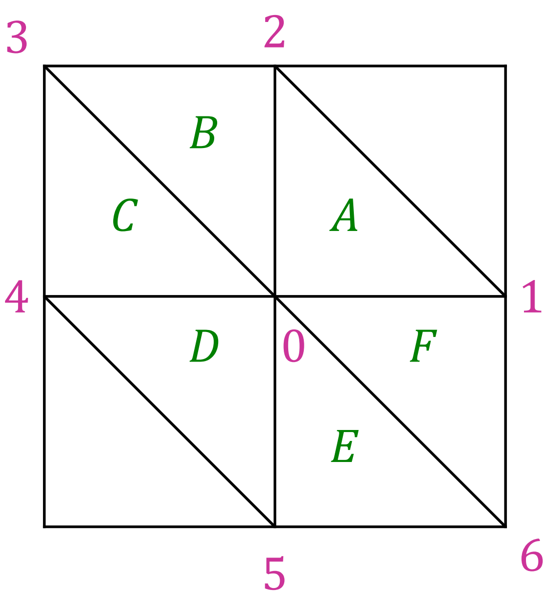
$$u^T(x, y) = \sum_{j \in T} u_j \varphi_j^T(x, y)$$

Step 1 : Multiply by a weight function, $\omega_i(x, y)$

$$\omega_i \nabla \cdot \vec{\mathfrak{F}} = 0$$

Step 2 : Integrate over a triangular element T

$$\iint_T \omega_i \nabla \cdot \vec{\mathfrak{F}} \, d\Omega = 0$$



$$\sum_{T \in \cup \Delta_i} \sum_{j \in T} \left\{ \iint_T \omega_i^T a \frac{\partial}{\partial x} \left(H_{zj} \varphi_j^T \right) d\Omega + \iint_T \omega_i^T a \frac{\partial}{\partial y} \left(E_{yj} \varphi_j^T \right) d\Omega \right\} = 0$$

$$\sum_{T \in \cup \Delta_i} \sum_{j \in T} \left\{ \iint_T \omega_i^T b \frac{\partial}{\partial x} \left(E_{yj} \varphi_j^T \right) d\Omega + \iint_T \omega_i^T b \frac{\partial}{\partial y} \left(H_{zj} \varphi_j^T \right) d\Omega \right\} = 0$$

$$\sum_{T \in \cup \Delta_i} \sum_{j \in T} \begin{bmatrix} a \dfrac{\partial \varphi_j^T}{\partial y} \iint_T \omega_i^T \, d\Omega & a \dfrac{\partial \varphi_j^T}{\partial x} \iint_T \omega_i^T \, d\Omega \\ b \dfrac{\partial \varphi_j^T}{\partial x} \iint_T \omega_i^T \, d\Omega & b \dfrac{\partial \varphi_j^T}{\partial y} \iint_T \omega_i^T \, d\Omega \end{bmatrix} \begin{pmatrix} E_{yj} \\ H_{zj} \end{pmatrix} = 0$$

$$\sum_{T \in \cup \Delta_i} \sum_{j \in T} \mathbb{M}_j^T \vec{U}_j = 0$$

$$\begin{aligned}\sum_{T \in \cup \Delta_i} \sum_{j \in T} \mathbb{M}_j^T \vec{U}_j = \{ & [\mathbb{M}_0^A \boxed{\vec{U}_0} + \mathbb{M}_1^A \vec{U}_1 + \mathbb{M}_2^A \boxed{\vec{U}_2}] \\ & + [\mathbb{M}_0^B \boxed{\vec{U}_0} + \mathbb{M}_2^B \boxed{\vec{U}_2} + \mathbb{M}_3^B \vec{U}_3] \\ & + [\mathbb{M}_0^C \boxed{\vec{U}_0} + \mathbb{M}_3^C \vec{U}_3 + \mathbb{M}_4^C \vec{U}_4] \\ & + [\mathbb{M}_0^D \boxed{\vec{U}_0} + \mathbb{M}_4^D \vec{U}_4 + \mathbb{M}_5^D \vec{U}_5] \\ & + [\mathbb{M}_0^E \boxed{\vec{U}_0} + \mathbb{M}_5^E \vec{U}_5 + \mathbb{M}_6^E \vec{U}_6] \\ & + [\mathbb{M}_0^F \boxed{\vec{U}_0} + \mathbb{M}_6^F \vec{U}_6 + \mathbb{M}_1^F \vec{U}_1]\} \quad = 0\end{aligned}$$

Likewise in the previous discussion, the *stiffness matrix* could be solved using Jacobi's iteration.

Overview of Finite-Element Method

Finite Element

Galerkin's Approach

$$\omega_i = \varphi_j$$

Galerkin's FE will be unstable in solving hyperbolic system as it is a central scheme.

Petrov-Galerkin's Approach

$$\omega_i \neq \varphi_j$$

SUPG (Streamline Upwind Petrov-Galerkin)

scalar

$$\omega_i = \varphi_j + \alpha\vec{\lambda}\cdot\nabla\varphi_i = \varphi_j + \alpha\frac{k_i^T}{S_T}$$

$$\alpha = C_1\frac{h}{\|\vec{\lambda}\|}$$

$$C_1 = 0.5$$

h is the typical length scale of the cell

Lax-Wendroff

if $\alpha = \Delta\tau/2$

Numerical Results

- Contour Plot
- L_2-errors

Numerical Set Up

$$a\frac{\partial H_z}{\partial x} + a\frac{\partial E_y}{\partial y} = 0$$
$$b\frac{\partial E_y}{\partial x} + b\frac{\partial H_z}{\partial y} = 0$$

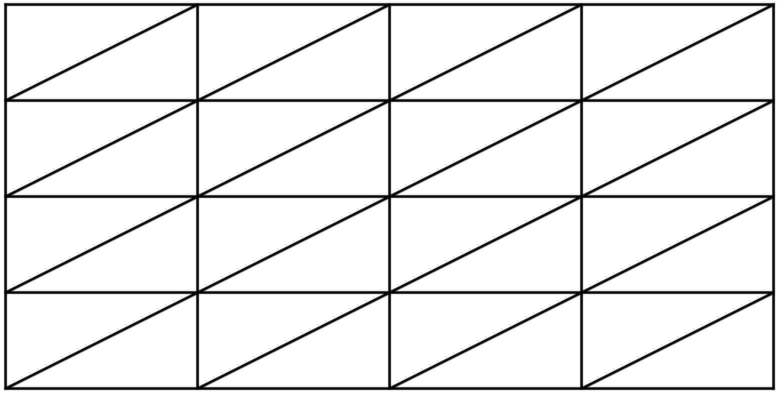
The scaled advection speed are $a = \left(\widetilde{H}_0/L\right)$ and $b = \left(\widetilde{H}_0 Z/L\right)$.

Setting $a = 1.0$, $b = 1.0$

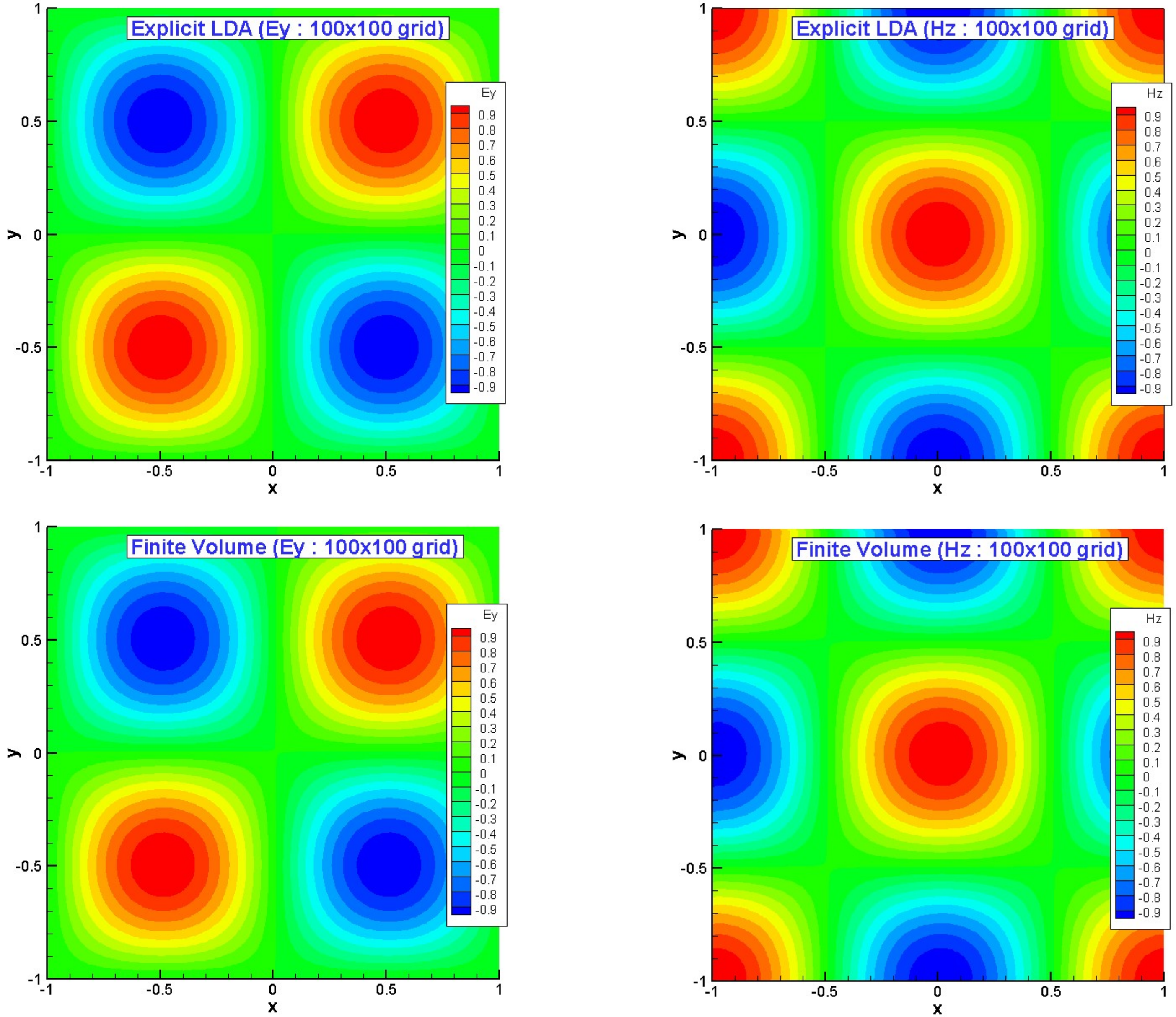
RR-Grid

To avoid the advection direction to fall along the diagonal of the mesh (where ***RD*** recovers its ***exact solution***), the skewness of the grid is set to be

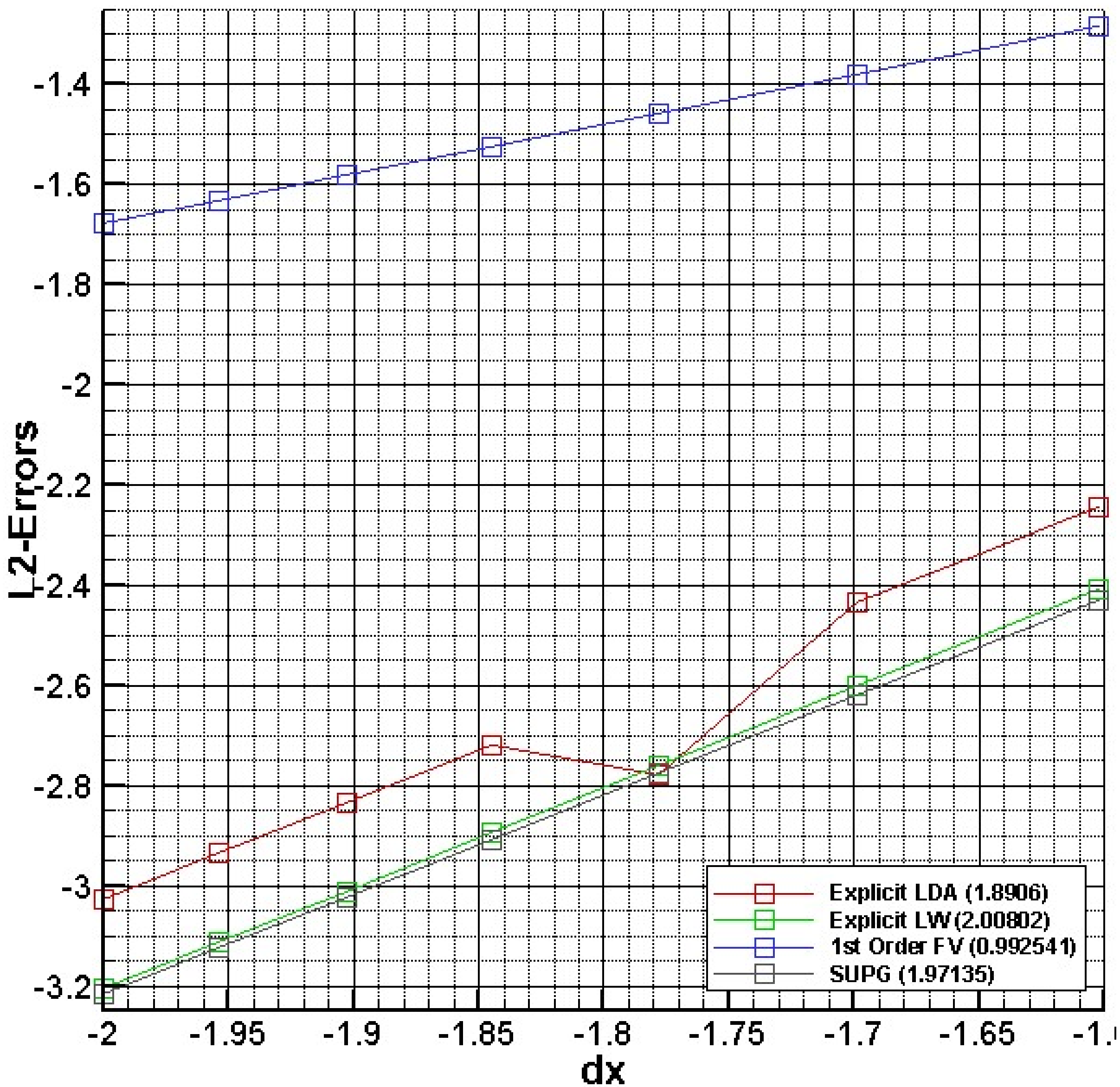
$$\Delta x : \Delta y = 2:1$$



Contour Plot



L_2-errors Plot



Conclusions

Schemes	Order-of-Accuracy
RD (LDA)	1.8906
RD (Lax-Wendroff)	2.00802
1st Order Finite Volume	0.992541
Finite Element (Galerkin's)	Diverging
Finite Element (SUPG)	1.97135

- Inconsistency in dimension.
  by scaling the equation to its dimensionless form.
- Adding the pseudo-time to the equations might pose some issues.
  use Jacobi's iteration
- Galerkin's finite-element method seems to work.
  the solution by Jacobi's iteration shows that Galerkin's approach fails. SUPG works on the other hand.