Computational Electromagnetic & Residual Distribution

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The Overview of Computational Electromagnetic (CEM)

Governing Equations – Maxwell's Equations

Static

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$ (Gauss Law) 1

$$\nabla \cdot \vec{H} = 0$$ 2

Dynamic

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$ (Faraday's Law) 3

$$\nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$ (Amperè's Law with Maxwell's correction) 4

The Hyperbolic Maxwell's Equations

Dynamic

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{H}}{\partial t}$$

TE mode

$$\frac{\partial E_x}{\partial t} - \frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y}\right) = 0$$

$$\frac{\partial E_y}{\partial t} - \frac{1}{\varepsilon}\left(-\frac{\partial H_z}{\partial x}\right) = 0$$

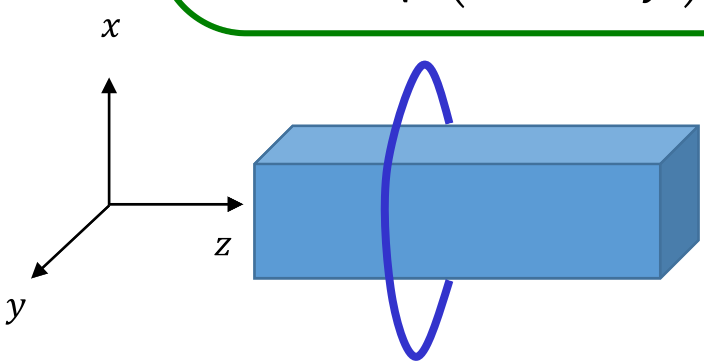
$$\frac{\partial H_z}{\partial t} + \frac{1}{\mu}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = 0$$

TM mode

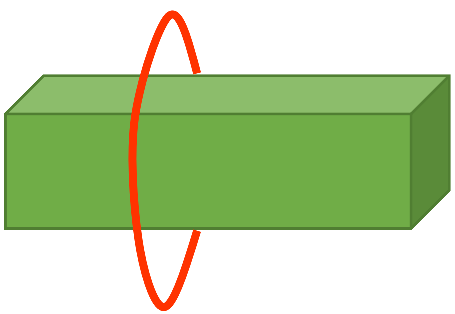
$$\frac{\partial H_x}{\partial t} - \frac{1}{\varepsilon}\left(-\frac{\partial E_z}{\partial y}\right) = 0$$

$$\frac{\partial H_y}{\partial t} - \frac{1}{\varepsilon}\left(\frac{\partial E_z}{\partial x}\right) = 0$$

$$\frac{\partial E_z}{\partial t} - \frac{1}{\mu}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = 0$$



Tranverse Electric (TE) : $E_z = 0$



Tranverse Magnetic (TM) : $H_z = 0$

Characteristic Speed

$$\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{\mathcal{F}} = 0$$

$$\frac{\partial \vec{U}}{\partial t} + \vec{\mathcal{A}} \cdot \nabla \vec{U} = 0$$

TM mode

$$\mathcal{A}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\mu} \\ 0 & -\frac{1}{\varepsilon} & 0 \end{pmatrix}$$

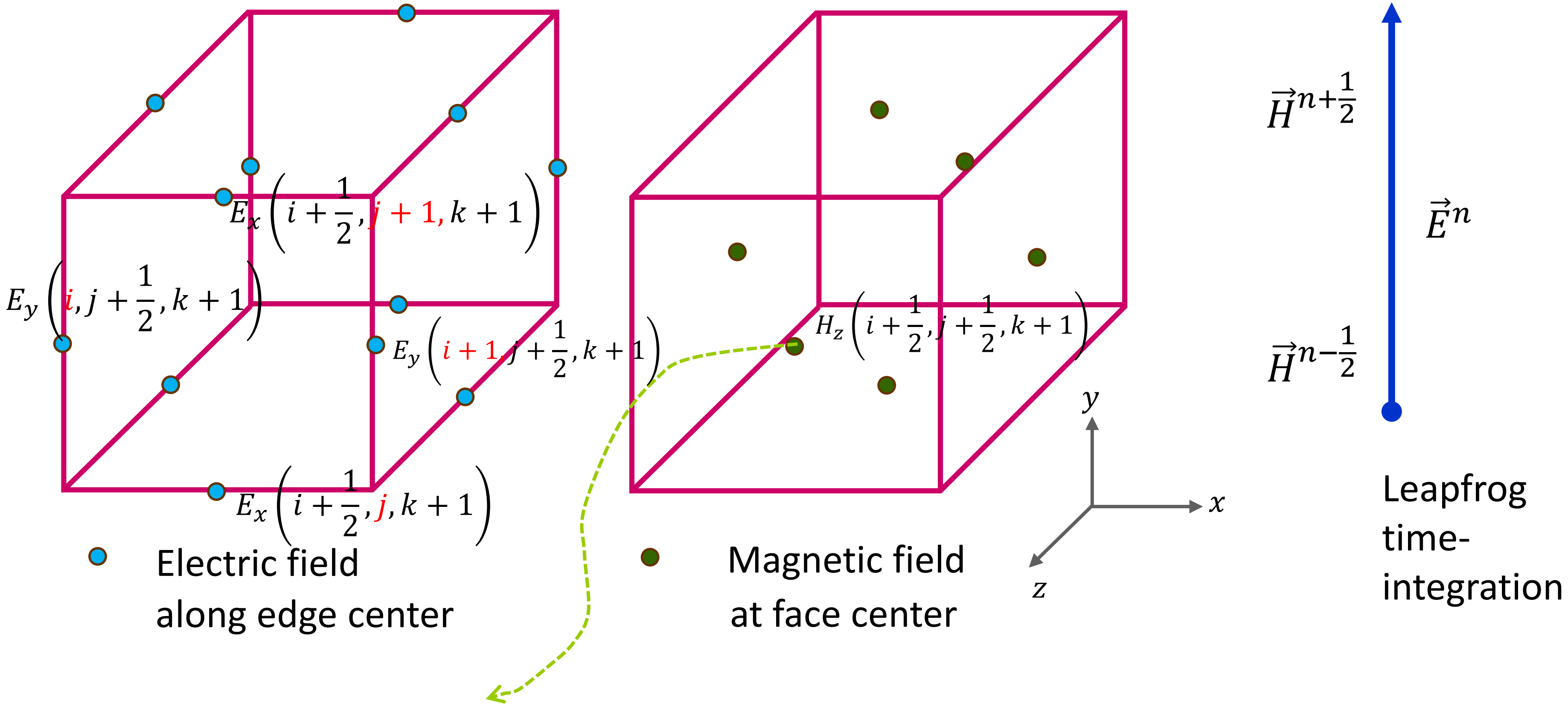
$$\mathcal{A}_y = \begin{pmatrix} 0 & 0 & \frac{1}{\mu} \\ 0 & 0 & 0 \\ \frac{1}{\varepsilon} & 0 & 0 \end{pmatrix}$$

TE mode

$$\mathcal{A}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\varepsilon} \\ 0 & \frac{1}{\mu} & 0 \end{pmatrix}$$

$$\mathcal{A}_y = \begin{pmatrix} 0 & 0 & -\frac{1}{\varepsilon} \\ 0 & 0 & 0 \\ -\frac{1}{\mu} & 0 & 0 \end{pmatrix}$$

(1) Finite-Difference Time-Domain (FDTD) – K. S. Yee (1966)



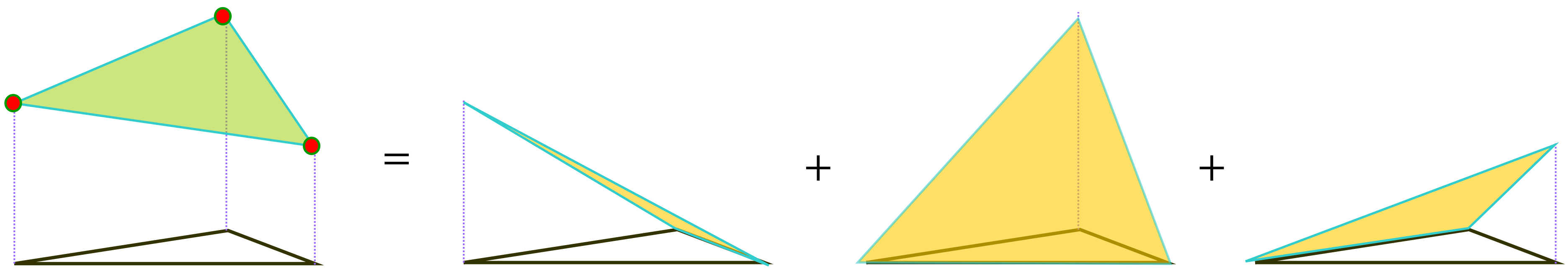
$$\frac{H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j+\frac{1}{2}, k+1\right)-H_z^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j+\frac{1}{2}, k+1\right)}{\Delta t}$$

$$=-\frac{1}{\mu}\left[\frac{E_y^n\left(i+1, j+\frac{1}{2}, k+1\right)-E_y^n\left(i, j+\frac{1}{2}, k+1\right)}{\Delta x}-\frac{E_x^n\left(i+\frac{1}{2}, j+1, k+1\right)-E_x^n\left(i+\frac{1}{2}, j, k+1\right)}{\Delta y}\right]$$

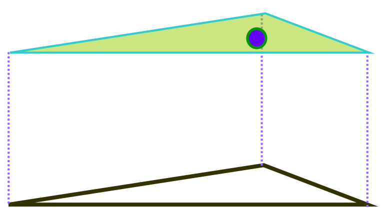
The information of E_x, E_y and E_z would not collocate on the same points..

(2) Finite-Element

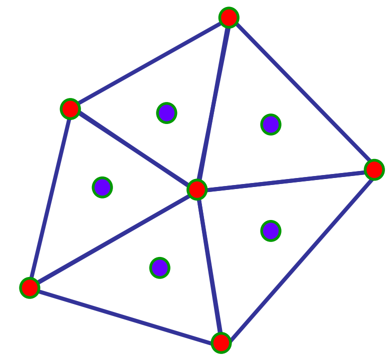
Nodal element *A. C. Cangellaris et al (1987)*



$$\vec{E}_h(x,y,t) = \sum_{j \in T_e} \vec{E}_j(t)\psi_j(x,y)$$



$$\vec{H}_h(x,y,t) = \sum_{j \in T_h} \vec{H}_j(t)\phi_j(x,y)$$

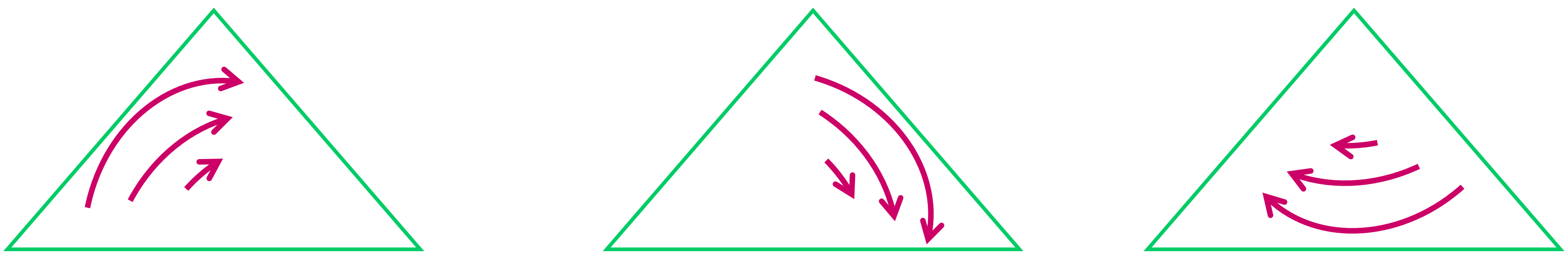


Every electric field node is contained within the volume of an element in the magnetic field grid, and every magnetic field node is contained within the volume of an element in the electric field grid.

Nédélev element (1981) / edge element / vector element

in electrodynamics by A. Bossavit and I. Mayergoyz (1989)

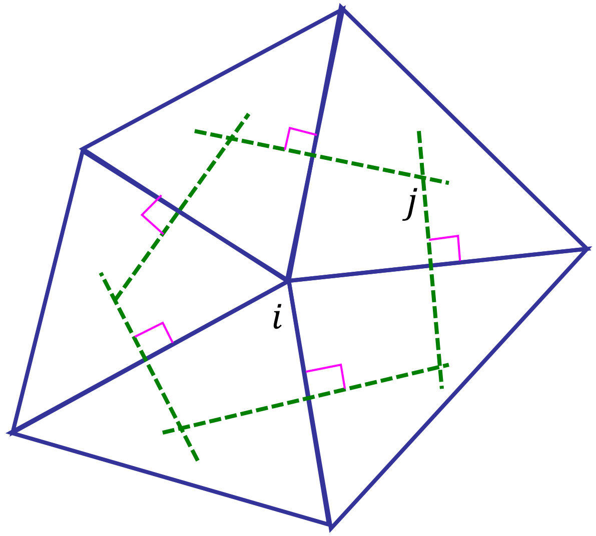
$$\vec{E}_h(x, y, t) = \sum_{j \in T} E_j(t) \vec{N}_j(x, y)$$



$\vec{N}_1(x, y)$ $\vec{N}_2(x, y)$ $\vec{N}_3(x, y)$

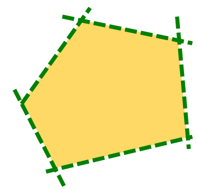
(3) Finite-Volume Time-Domain

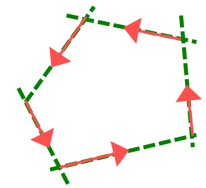
by F. Hermerline (1993)



(1)

$$\frac{\partial}{\partial t}\left(\int_{V_i} B\right) + \sum_{k=1}^{N_i} \int_{\partial V_i^k} \vec{E}\cdot\vec{\tau}_i^k = 0$$

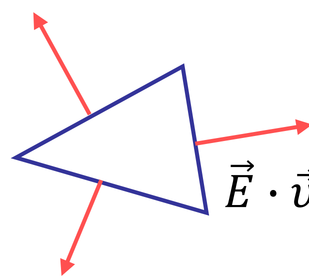




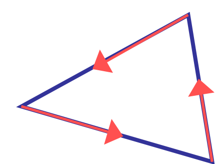
$\vec{E}\cdot\vec{\tau}_i^k$

(2)

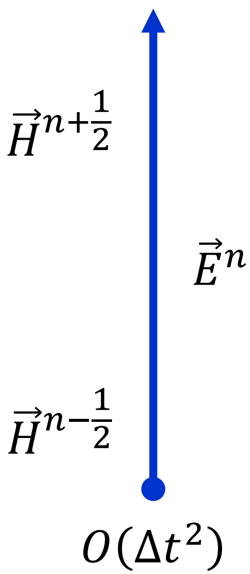
$$\frac{\partial}{\partial t}\left(\int_{\partial D_j^l} \vec{E}\cdot\vec{v}_j^l\right) - c^2 \int_{\partial D_j^l} \nabla B\cdot\vec{\tau}_j^l = 0$$



$\vec{E}\cdot\vec{v}_j^l$



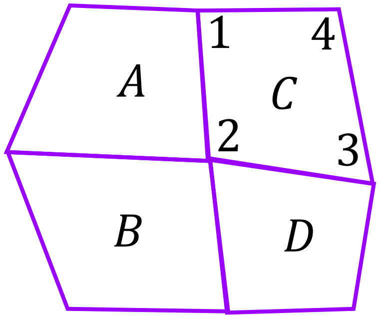
$\nabla B\cdot\vec{\tau}_j^l$



(3)

$$\sum_{i=1}^{N_j} \int_{\partial D_j^l} \vec{E}\cdot\vec{v}_j^l = \frac{1}{\varepsilon}\int_{D_j} \rho$$

by Deore and Chatterjee (2010)



Interchanging of time-derivative with spatial derivative

$$\delta\vec{U} = \left(\frac{\partial \vec{U}}{\partial t}\right)^n \Delta t + \left(\frac{\partial^2 \vec{U}}{\partial t^2}\right)^n \frac{\Delta t^2}{2} + O(\Delta t^3)$$

$$\delta\vec{U} = -\left(\frac{\partial \vec{f}}{\partial x} + \frac{\partial \vec{g}}{\partial y}\right)^n \Delta t + O(\Delta t^2)$$

Discrete numerical flux

$$\Delta\vec{U}_C = \frac{\Delta t}{S_T}\left(\sum_{j=1}^{4} \vec{F}_j \cdot \vec{n}_j\right)$$

Unsteady fluxes

$$\Delta\vec{F}_C = \frac{\partial \vec{f}}{\partial \vec{U}}\Delta\vec{U}_C \qquad \Delta\vec{G}_C = \frac{\partial \vec{g}}{\partial \vec{U}}\Delta\vec{U}_C$$

Second-order correction

$$\Delta\vec{f}_C = \frac{\Delta t}{S_T}\left(\Delta\vec{F}_C \Delta y^l + \Delta\vec{G}_C \Delta x^l\right) \qquad \Delta\vec{g}_C = \frac{\Delta t}{S_T}\left(\Delta\vec{F}_C \Delta y^m + \Delta\vec{G}_C \Delta x^m\right)$$

Implicit Nodal Update

$$\left(\delta\vec{U}_1\right)_C = \frac{1}{4}\left[\Delta\vec{U}_C - \Delta\vec{f}_C - \Delta\vec{g}_C\right]$$

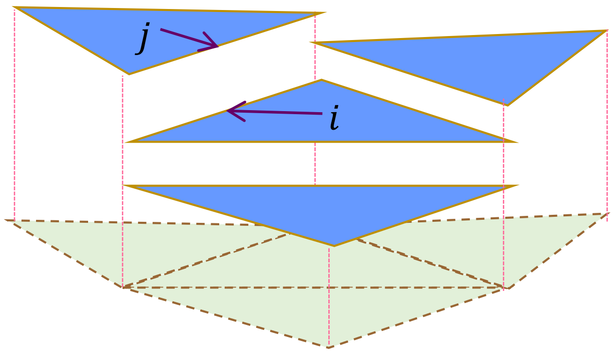
$$\left(\delta\vec{U}_2\right)_C = \frac{1}{4}\left[\Delta\vec{U}_C - \Delta\vec{f}_C + \Delta\vec{g}_C\right]$$

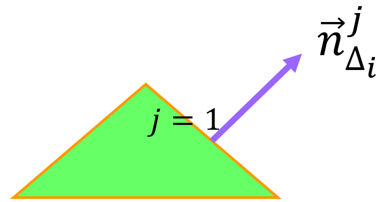
$$\left(\delta\vec{U}_3\right)_C = \frac{1}{4}\left[\Delta\vec{U}_C + \Delta\vec{f}_C + \Delta\vec{g}_C\right]$$

$$\left(\delta\vec{U}_4\right)_C = \frac{1}{4}\left[\Delta\vec{U}_C + \Delta\vec{f}_C - \Delta\vec{g}_C\right]$$

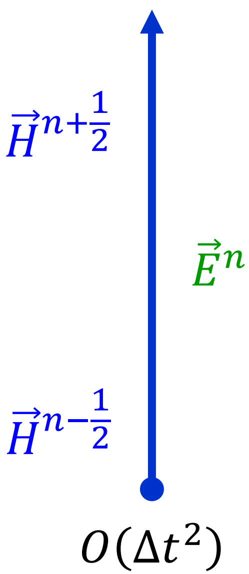
$$\delta\vec{U}_1 = \sum_{m=1}^{4}\left(\delta\vec{U}_1\right)_m$$

by Remaki (1999), S. Piperno and Remaki (2002)





$$\vec{\mathcal{F}}_i = \frac{\vec{E}_i^n + \vec{E}_j^n}{2}$$



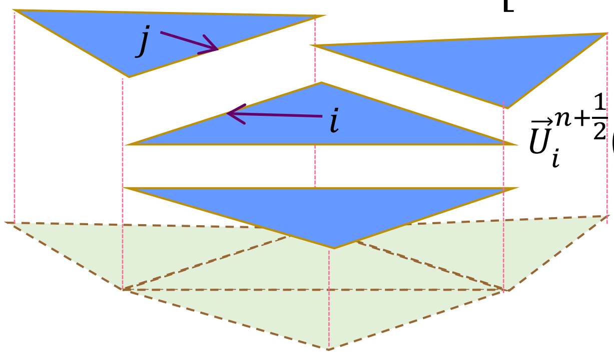
$$S_{\Delta_i}\frac{\vec{H}_i^{n+\frac{1}{2}} - \vec{H}_i^{n-\frac{1}{2}}}{\Delta t} + \sum_{j=1}^{3} n_{\Delta_i}^{j}\vec{\mathcal{F}}_i\left(\frac{\vec{E}_i^n + \vec{E}_j^n}{2}\right) = 0$$

$$S_{\Delta_i}\frac{\vec{E}_i^n - \vec{E}_i^{n-1}}{\Delta t} + \sum_{j=1}^{3} n_{\Delta_i}^{j}\vec{\mathcal{F}}_i\left(\frac{\vec{H}_i^{n-\frac{1}{2}} + \vec{H}_j^{n-\frac{1}{2}}}{2}\right) = 0$$

$$\vec{\mathcal{F}}_i = \frac{\vec{H}_i^{n-\frac{1}{2}} + \vec{H}_j^{n-\frac{1}{2}}}{2}$$

by T. Z. Ismagilov (2016)

$$\vec{U}_j^{n+\frac{1}{2}}(\vec{X}^\Gamma) = \vec{U}_j^n + \frac{\partial \vec{U}_j^n}{\partial \vec{x}}(\vec{X}^\Gamma - \vec{X}_j) - \frac{\Delta t}{2}\left[\mathcal{A}_x \frac{\partial \vec{U}_j^n}{\partial x} + \mathcal{A}_y \frac{\partial \vec{U}_j^n}{\partial y}\right]$$



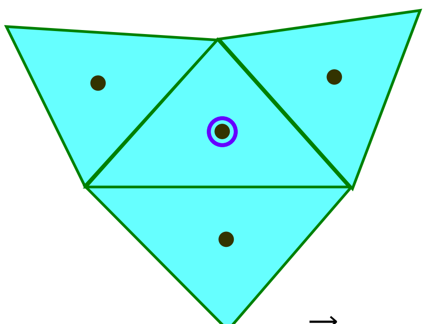
Lax-Wendroff time-integration

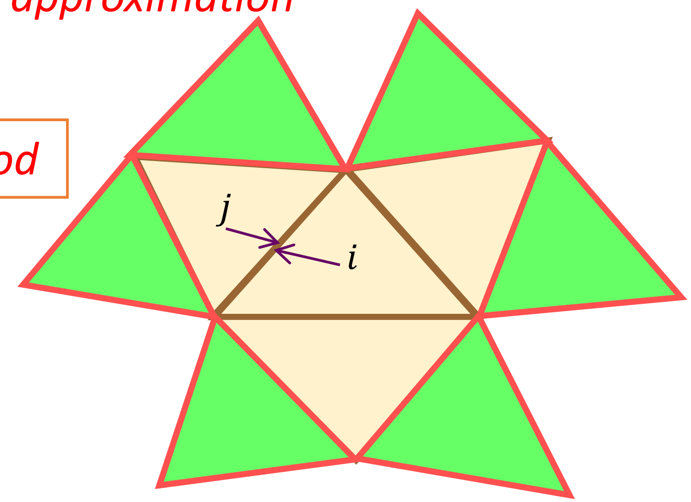
$$\vec{U}_i^{n+\frac{1}{2}}(\vec{X}^\Gamma) = \vec{U}_i^n + \frac{\partial \vec{U}_i^n}{\partial \vec{x}}(\vec{X}^\Gamma - \vec{X}_i) - \frac{\Delta t}{2}\left[\mathcal{A}_x \frac{\partial \vec{U}_i^n}{\partial x} + \mathcal{A}_y \frac{\partial \vec{U}_i^n}{\partial y}\right]$$

Van Leer's spatial approximation

Gradient calculation

Godunov Method





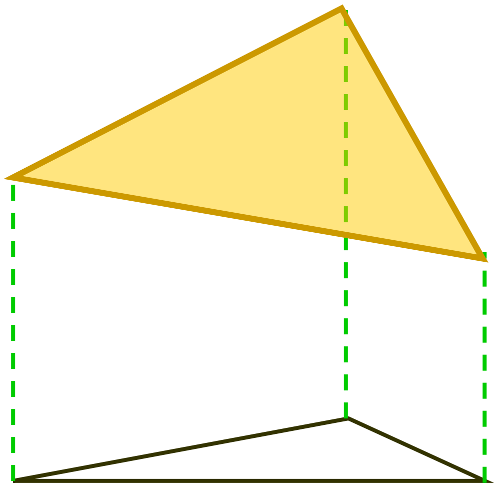
$$\frac{\partial \vec{U}^n}{\partial \vec{x}} = (\vec{U}_1, \cdots, \vec{U}_m)\vec{X}^T\left(\vec{X}\vec{X}^T\right)^{-1}$$

$$S_{\Delta_i}\frac{\vec{U}_i^{n+1} - \vec{U}_i^n}{\Delta t} + \sum_{j=1}^{3} n_{\Delta_i}^j \vec{\mathcal{F}}_i^j = 0$$

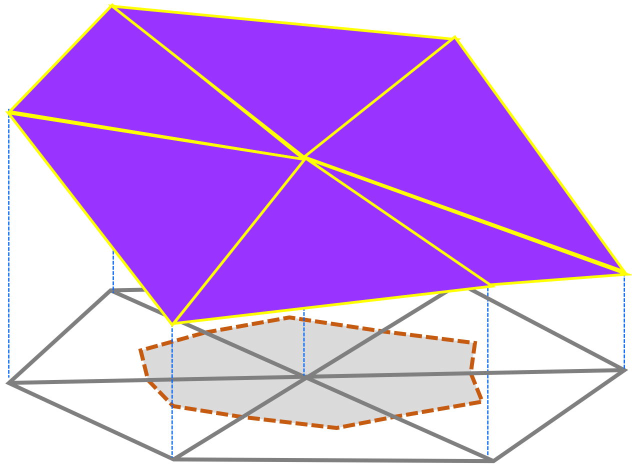
$$\vec{\mathcal{F}} = \mathcal{A}^+\vec{U}_i + \mathcal{A}^-\vec{U}_j$$

Residual Distribution Method in Maxwell’s Equations

Lagrange's Interpolation Function



$$\vec{U}_h(x,y,t) = \sum_{j\in T} \vec{U}_j(t)\psi_j(x,y)$$



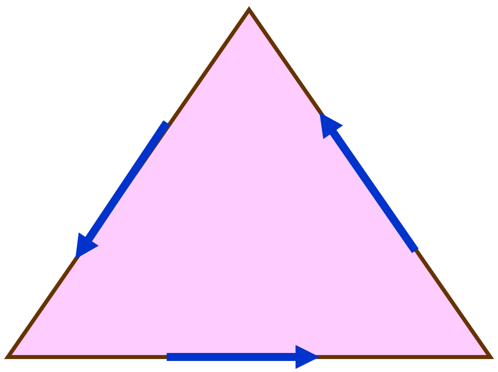
$$\nabla\psi_j(x,y) = \frac{\vec{n}_j}{2S_T}$$

$$\nabla\vec{U}_h(x,y,t) = \sum_{j\in T} \vec{U}_j(t)\nabla\psi_j(x,y) = \sum_{j\in T} \frac{\vec{n}_j}{2S_T}\vec{U}_j(t)$$

Flux Residual

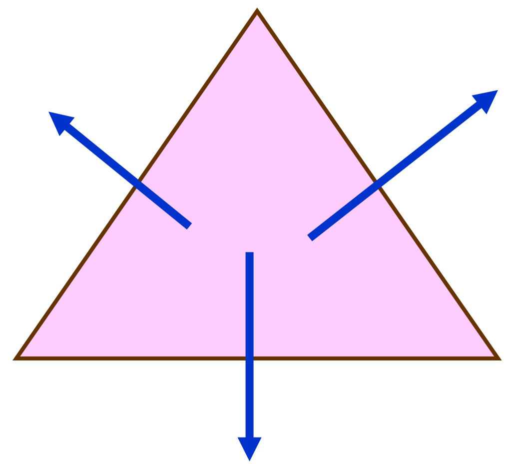
Contour Integration

$$\vec{\Phi}^{\,T} = \oint_{\partial T} \vec{\mathcal{F}} \cdot d\hat{l} \cong \frac{1}{2} \sum_{j \in T} \vec{\mathcal{F}}_j \cdot \vec{n}_j$$



Volume Integration

$$\vec{\Phi}^{\,T} = \iint_{T} \nabla \cdot \vec{\mathcal{F}} d\Omega \cong \sum_{j \in T} K_j \vec{U}_j$$



Explicit scheme

$$\vec{U}_i^{n+1} = \vec{U}_i^{n} - \frac{\Delta t}{S_i} \sum_{T \in \cup \Delta_i} B_i^T \vec{\Phi}_j^T$$

Implicit scheme

$$\sum_{T \in \cup \Delta_i} \left[\sum_{j \in T} M_{ij}^T \left(\frac{d\vec{U}}{dt} \right)_j + B_i^T \vec{\Phi}^T \right] = 0 \Rightarrow \frac{\delta \vec{U}}{\delta \tau} + \sum_{T \in \cup \Delta_i} \left[\sum_{j \in T} M_{ij}^T \left(\frac{d\vec{U}}{dt} \right)_j + B_i^T \vec{\Phi}^T \right] = 0$$

Flux Splitting

TM mode

$$K_j = \frac{|\vec{n}_j|}{2}\begin{pmatrix} 0 & 0 & \frac{\eta_{jy}}{\mu} \\ 0 & 0 & -\frac{\eta_{jx}}{\mu} \\ \frac{\eta_{jy}}{\varepsilon} & -\frac{\eta_{jx}}{\varepsilon} & 0 \end{pmatrix}$$

$$K_j^{\pm} = \frac{|\vec{n}_j|}{4}\begin{pmatrix} \pm c\eta_{jy}^2 & \mp c\eta_{jx}\eta_{jy} & \frac{c\eta_{jy}}{Z} \\ \mp c\eta_{jx}\eta_{jy} & \pm c\eta_{jx}^2 & -\frac{c\eta_{jx}}{Z} \\ c\eta_{jy}Z & -c\eta_{jx}Z & \pm c \end{pmatrix}$$

TE mode

$$K_j = \frac{|\vec{n}_j|}{2}\begin{pmatrix} 0 & 0 & -\frac{\eta_{jy}}{\varepsilon} \\ 0 & 0 & \frac{\eta_{jx}}{\varepsilon} \\ -\frac{\eta_{jy}}{\mu} & \frac{\eta_{jx}}{\mu} & 0 \end{pmatrix}$$

$$K_j^{\pm} = \frac{|\vec{n}_j|}{4}\begin{pmatrix} \pm c\eta_{jy}^2 & \mp c\eta_{jx}\eta_{jy} & -c\eta_{jy}Z \\ \mp c\eta_{jx}\eta_{jy} & \pm c\eta_{jx}^2 & c\eta_{jx}Z \\ -\frac{c\eta_{jy}}{Z} & \frac{c\eta_{jx}}{Z} & \pm c \end{pmatrix}$$

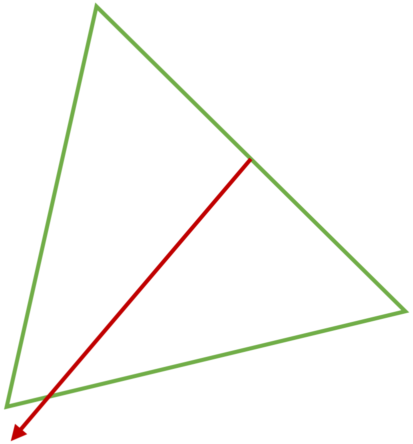
Diagonalization

$$K_j = \frac{|\vec{n}_j|}{2}\left(\mathcal{A}_{jx}\eta_{jx} + \mathcal{A}_{jy}\eta_{jy}\right) \qquad k_j = \frac{1}{2}\vec{\lambda}\cdot\vec{n}_j$$

$$K_j = \frac{|\vec{n}_j|}{2} R\Lambda R^{-1}$$

$$\Lambda = \begin{pmatrix} -c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$K_j^{\pm} = \frac{|\vec{n}_j|}{2} R\Lambda^{\pm} R^{-1}$$



$$\vec{n}_j = \eta_{jx}|\vec{n}_j|\hat{x} + \eta_{jy}|\vec{n}_j|\hat{y}$$

RD-Lax-Wendroff

Spatial flux residual

$$\left(\nabla \cdot \vec{\mathcal{F}}_h\right)^{n+\frac{1}{2}} = -\left(\frac{\partial \vec{U}_h}{\partial t}\right)^{n+\frac{1}{2}}$$

$$= -\left(\frac{\partial \vec{U}_h}{\partial t}\right)^{n} - \frac{\Delta t}{2}\left(\frac{\partial^2 \vec{U}_h}{\partial t^2}\right)^{n} + O(\Delta t^2)$$

$$= \left(\nabla \cdot \vec{\mathcal{F}}_h\right)^{n} + \frac{\Delta t}{2}\vec{\mathcal{A}} \cdot \nabla\left(\frac{\partial \vec{U}_h}{\partial t}\right)^{n} + O(\Delta t^2)$$

Time discretization

$$\left(\frac{\partial \vec{U}_i}{\partial t}\right)^{n+\frac{1}{2}} = \frac{\vec{U}_i^{n+1} - \vec{U}_i^{n}}{\Delta t} + O(\Delta t^2)$$

$$\vec{U}_i^{n+1} = \vec{U}_i^{n} - \frac{\Delta t}{S_i}\sum_{T\in\cup\Delta_i}\boxed{\left[\frac{1}{3}I + \frac{\Delta t}{2S_T}K_i^T\right]}\vec{\Phi}_j^T$$

Distribution matrix

RD-LDA

Implicit nodal update

$$\sum_{T \in \cup \Delta_i} \left[\sum_{j \in T} M_{ij}^T \left(\frac{d\vec{U}}{dt} \right)_j + B_i^T \vec{\Phi}^T \right] = 0$$

Mass matrix

$$M_{ij}^T = \frac{S_T}{4} \begin{bmatrix} 2B_1^T & B_1^T & B_1^T \\ B_2^T & 2B_2^T & B_2^T \\ B_3^T & B_3^T & 2B_3^T \end{bmatrix}$$

Distribution matrix

$$B_j^T = K_j^+ \left(\sum_{j \in T} K_j^+ \right)^{-1}$$

Where comes the mass-matrix?

$$\vec{U}_h(x,y,t) = \sum_{j\in T} \vec{U}_j(t)\psi_j(x,y)$$

$$\iint_T \omega_i I \left[\frac{\partial \vec{U}_h}{\partial t} + \nabla \cdot \vec{\mathcal{F}}_h\right] d\Omega = 0$$

I is the identity matrix.

$$\iint_T \omega_i I \left[\sum_{j\in T} \frac{\partial \vec{U}_j}{\partial t}\psi_j + \vec{\mathcal{A}} \cdot \sum_{j\in T} \vec{U}_j \nabla \psi_j\right] d\Omega = 0$$

$$\iint_T \omega_i I \left[\sum_{j\in T} \frac{\partial \vec{U}_j}{\partial t}\psi_j + \vec{\mathcal{A}} \cdot \sum_{j\in T} \vec{U}_j \nabla \psi_j\right] d\Omega = 0$$

$$B_i^T = \frac{1}{S_T} \iint_T \omega_i I d\Omega$$

$$\sum_{T\in\cup\Delta_i} \sum_{j\in T} \left(\frac{\partial \vec{U}_j}{\partial t} \underbrace{\iint_T \omega_i I \psi_j I \, d\Omega}_{M_{ij}^T}\right) + \sum_{T\in\cup\Delta_i} B_i^T \vec{\Phi}^T = 0$$

What can we do with the mass matrix?

The Lax-Wendroff scheme if viewed from the FE perspective, must also contains mass-matrix.

$$\sum_{T\in\cup\Delta_i} S_T \frac{I}{12}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix} + \sum_{T\in\cup\Delta_i} B_i^T \vec{\Phi}^T = 0$$

$$\vec{U}_i^{n+1} = \vec{U}_i^n - \frac{\Delta t}{S_i}\sum_{T\in\cup\Delta_i}\left[\frac{1}{3}I + \frac{\Delta t}{2S_T}K_i^T\right]\vec{\Phi}_j^T$$

Row-Mass-Lumping

Applying the same row-mass-lumping concept to the upwind LDA scheme makes the scheme explicit.

$$\sum_{T\in\cup\Delta_i}\left[\sum_{j\in T} M_{ij}^T \left(\frac{d\vec{U}}{dt}\right)_j + B_i^T\vec{\Phi}^T\right] = 0$$

$$\left(\sum_{T\in\cup\Delta_i} S_T B_i^T\right)\frac{\Delta\vec{U}_i}{\Delta t} + \sum_{T\in\cup\Delta_i} B_i^T\vec{\Phi}^T = 0$$

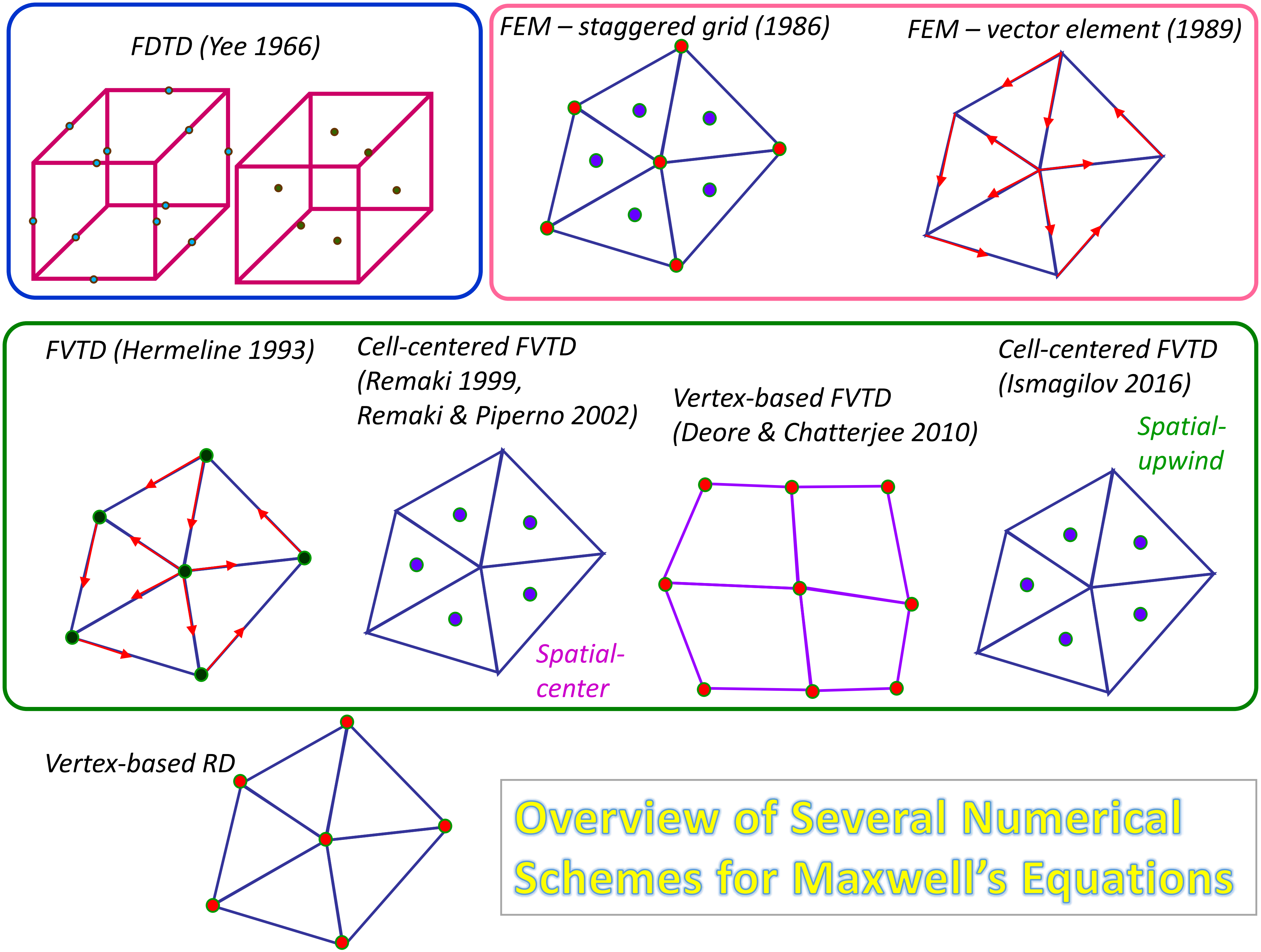
Second-order Time-Integration

$$\left(\sum_{T\in\cup\Delta_i} S_T B_i^T\right)\frac{\vec{U}_i^{n+1}-\vec{U}_i^{n-1}}{\Delta t}+\sum_{T\in\cup\Delta_i} B_i^T\vec{\Phi}^T\left(\vec{U}_i^n\right)=0$$

Two-Stage Runge-Kutta Time-Integration

$$\left(\sum_{T\in\cup\Delta_i} S_T B_i^T\right)\frac{\vec{U}_i^{n+\frac{1}{2}}-\vec{U}_i^{n}}{\Delta t/2}+\sum_{T\in\cup\Delta_i} B_i^T\vec{\Phi}^T\left(\vec{U}_i^n\right)=0$$

$$\left(\sum_{T\in\cup\Delta_i} S_T B_i^T\right)\frac{\vec{U}_i^{n+1}-\vec{U}_i^{n}}{\Delta t}+\sum_{T\in\cup\Delta_i} B_i^T\vec{\Phi}^T\left(\vec{U}_i^{n+\frac{1}{2}}\right)=0$$


FDTD (Yee 1966)
FEM – staggered grid (1986)
FEM – vector element (1989)
FVTD (Hermeline 1993)
Cell-centered FVTD
(Remaki 1999,
Remaki & Piperno 2002)
Spatial-
center
Vertex-based FVTD
(Deore & Chatterjee 2010)
Cell-centered FVTD
(Ismagilov 2016)
Spatial-
upwind
Vertex-based RD
Overview of Several Numerical Schemes for Maxwell's Equations