Basic Residual Distribution: from comparison with FE and FV to 3D RD construction

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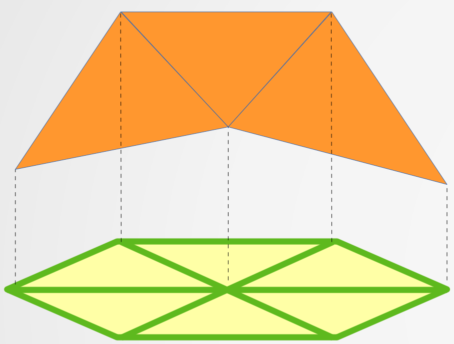
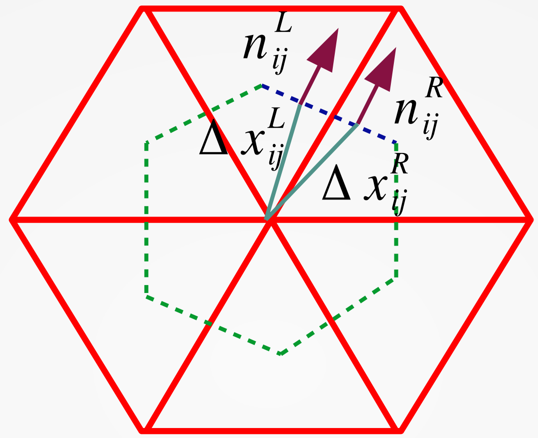
FE, FV & RD for Steady Case

Finite Element Method



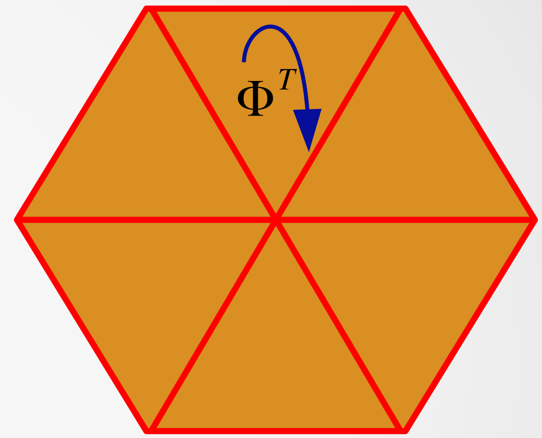
Elliptic and parabolic PDE

Finite Volume Method



Parabolic and hyperbolic PDE

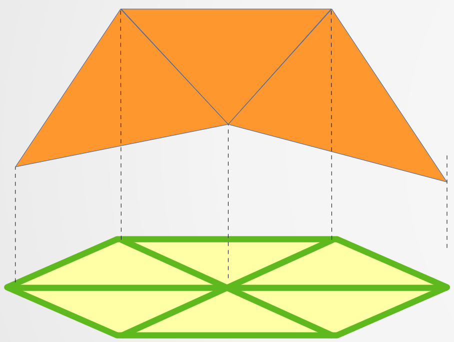
Residual Distribution Method



Parabolic and hyperbolic PDE

Elliptic PDE	$\nabla^2 u = 0$	*Laplace equation*
Parabolic PDE	$\frac{\partial u}{\partial t} = \alpha\left(\nabla^2 u\right)$	*Heat transfer equation, diffusion equation*
Hyperbolic PDE	$\frac{\partial u}{\partial t} + \nabla \cdot F = \frac{\partial u}{\partial t} + \lambda \cdot \nabla u = 0$	*wave equation* $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$

Finite Element



$$u_h^T = \sum_{j \in T} \psi^T(x, y) u_j$$

- Step 1 : Multiply the governing equation with a weight function and integrate over the triangular element. Using weak formulation and integration by parts to reduce the order of partial differential equation.

$$\iint_T \psi_i \left(\frac{\partial u_h}{\partial t} - \nabla^2 u_h\right) d\Omega = 0$$

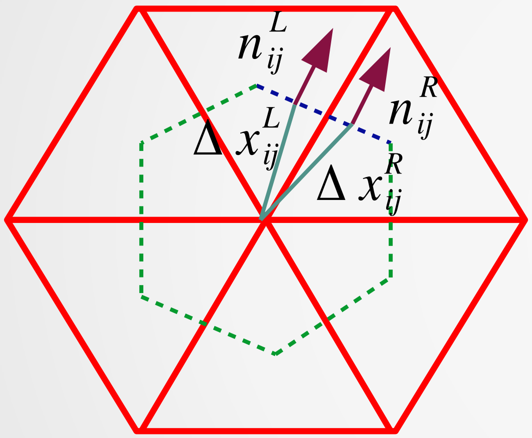
$$\sum_{j \in T} \left[\frac{du_j}{dt} \iint_T \psi_i \psi_j \, d\Omega + (\nabla \psi_i \cdot \nabla \psi_j) u_j \iint_T d\Omega\right] = 0$$

- Step 2 : Assembly all the elements and form *n* equations for *n* nodes in the domain.
- Step 3 : Solve the simultaneous system of equation.

<u>Bonus</u>

- Connect all the approximate value at intersection points using Lagrange interpolation

Finite Volume



- Step 1 : Create triangular primary elements and median dual cell.
- Step 2 : Emphasize on median dual cell. Equate the flux leaving the median dual cell is balanced out by the flux entering the median dual cell.

Stopping criteria :

flux leaving = flux entering

First-order accurate scheme

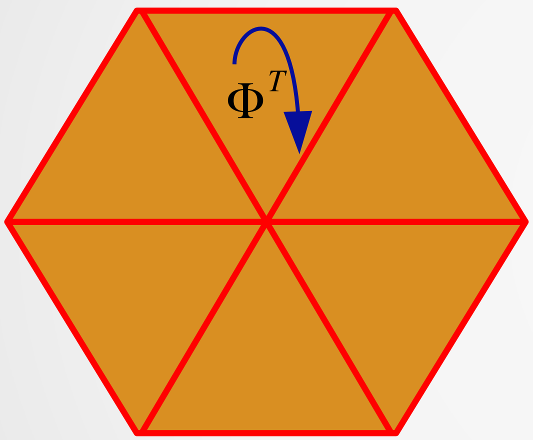
$$u_i^{k+1} = u_i^k - \frac{\Delta t}{S_i} \sum_{k \in k_j} \left(F_i^+ \cdot n_{ij}^L + F_i^+ \cdot n_{ij}^R \right)$$

Second-order accurate scheme

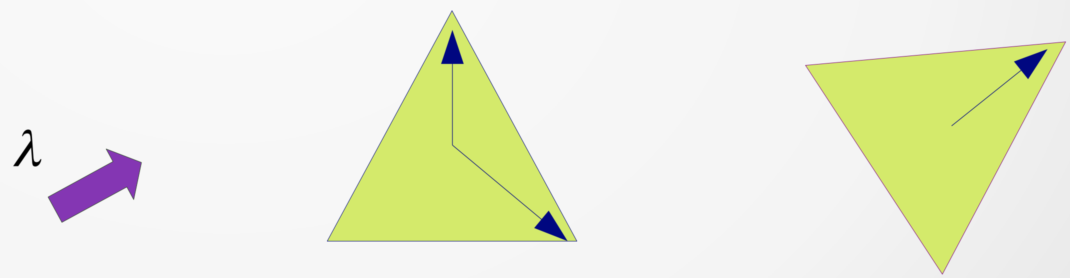
$$u_i^{k+1} = u_i^k - \frac{\Delta t}{S_i} \sum_{k \in k_j} \left(\left(F_i^+ + \nabla F_i \cdot \Delta x_{ij}^L \right) \cdot n_{ij}^L + \left(F_i^+ + \nabla F_i \cdot \Delta x_{ij}^R \right) \cdot n_{ij}^R \right)$$

Residual Distribution

$$\Phi^T = \iint_T \nabla \cdot F \, d\Omega = \oint_{\partial T} F \cdot dl$$



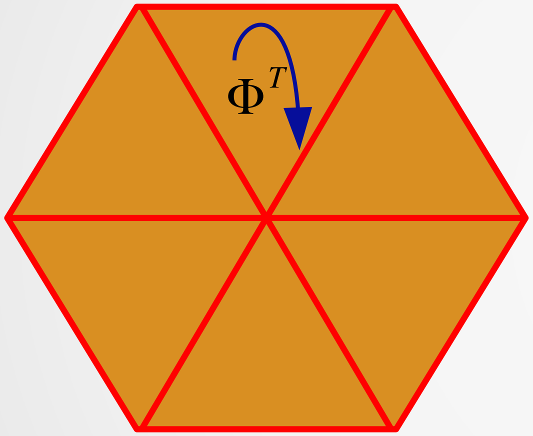
- Step 1 : Create triangular primary elements.
- Step 2 : Calculate flux residual or flux fluctuation Φ^T
- Step 3 : Distribute the residual or split the fluctuation according to characteristic speed.



<u>Driving mechanism :</u>
If the residual is not zero, just distribute it.

<u>Stopping criteria :</u> $\Phi^T \rightarrow 0$

Residual Distribution for Time-Dependent



Steady Case $\nabla \cdot F = 0$

$$\Phi^T = \iint_T \nabla \cdot F \, d\Omega = \oint_{\partial T} F \cdot dl \rightarrow 0$$

Driving the residual within each element T towards zero.
Eventually, the fluctuation approaches zero.

Unsteady Case $\frac{\partial u}{\partial t} + \nabla \cdot F = 0$

$$\Phi^T = \iint_T \nabla \cdot F \, d\Omega = \oint_{\partial T} F \cdot dl \rightarrow -\left(\frac{S_T}{3}\right) \sum_{j \in T} \left(\frac{\partial u_j}{\partial t}\right)$$

The residual at every single time step is no longer be zero, but some other values.

Time Discretization

Therefore, in time-dependent cases, the spatial flux has to be evaluated, and then updated with proper time-marching scheme.

First-order accurate

Forward-Euler

$$\left(\frac{du}{dt}\right)^{n} \approx \frac{u^{n+1}-u^{n}}{\Delta t}+O(\Delta t) \qquad \frac{du}{dt}=-\frac{\Delta t}{S_i}\sum_{T\in\cup\Delta_i}\beta_i^T\Phi^T(u^n)$$

Second-order accurate

Backward-time

$$\left(\frac{du}{dt}\right)^{n+1} \approx \frac{3u^{n+1}-4u^{n}+u^{n-1}}{2\Delta t}+O(\Delta t^2) \qquad \frac{du}{dt}=-\frac{\Delta t}{S_i}\sum_{T\in\cup\Delta_i}\beta_i^T\Phi^T(u^{n+1})$$

Leapfrog-time

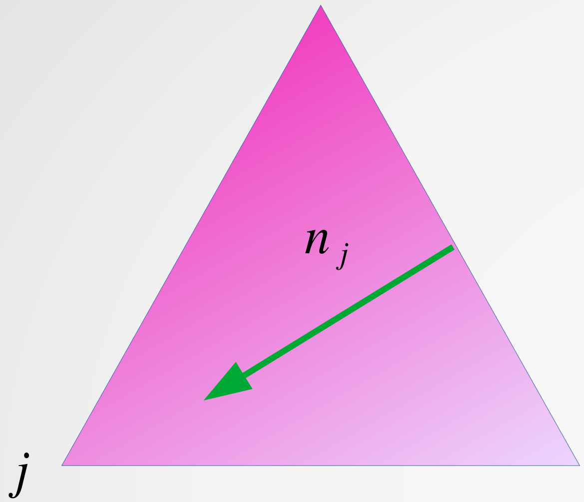
$$\left(\frac{du}{dt}\right)^{n} \approx \frac{u^{n+1}-u^{n-1}}{2\Delta t}+O(\Delta t^2) \qquad \frac{du}{dt}=-\frac{\Delta t}{S_i}\sum_{T\in\cup\Delta_i}\beta_i^T\Phi^T(u^n)$$

Predictor-corrector (RK2)

$$\left(\frac{du}{dt}\right)^{n+\frac{1}{2}} \approx \frac{u^{n+1}-u^{n}}{\Delta t}+O(\Delta t^2) \qquad u^{n+\frac{1}{2}}=u^n-\frac{\Delta t/2}{S_i}\sum_{T\in\cup\Delta_i}\beta_i^T\Phi^T(u^n)$$

$$\frac{du}{dt}=-\frac{\Delta t}{S_i}\sum_{T\in\cup\Delta_i}\beta_i^T\Phi^T(u^{n+\frac{1}{2}})$$

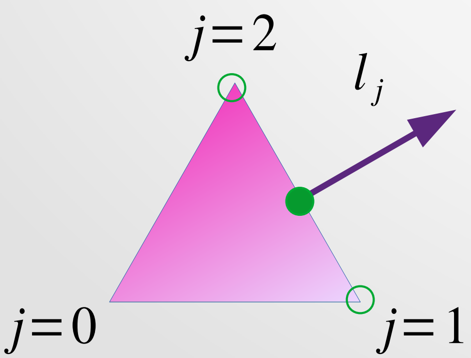
Flux Residual



The flux residual is

$$\Phi^T = \sum_{j \in T} k_j u_j = \sum_{j \in T} (\frac{1}{2} \lambda \cdot n_j) u_j$$

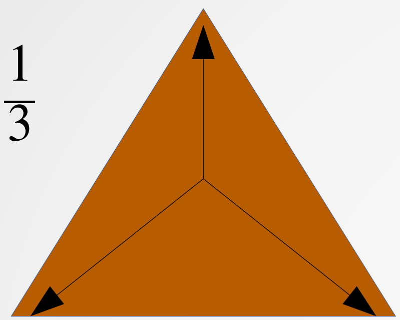
The convention of inward scaled normal in RD to calculate flux residual is equivalent to evaluating the average of the flux across every single edge..



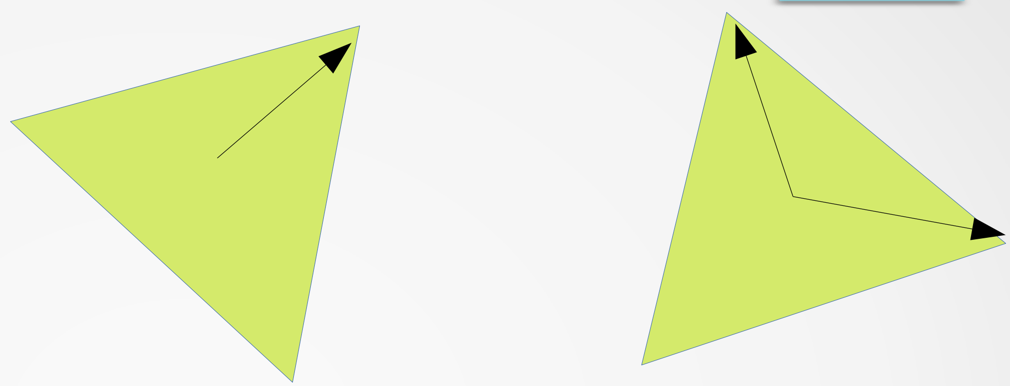
$$\Phi^T = \iint_T \nabla \cdot F = \oint_{\partial T} F \cdot dl \approx \sum_{j \in T} \frac{(F_{j+1} + F_{j-1})}{2} \cdot l_j = \sum_{j \in T} k_j u_j$$

Distribution Coefficient & Temporal Update

$$\beta_j = \frac{1}{3}$$



Space-centered
(Galerkin type)



Upwind LDA

$$\beta_j = \frac{k_j^+}{\sum_{j \in T} k_j^+}$$

RK2

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t/2}{S_i} \sum_{T \in \cup \Delta_i} \frac{1}{3} \Phi^T(u^n)$$

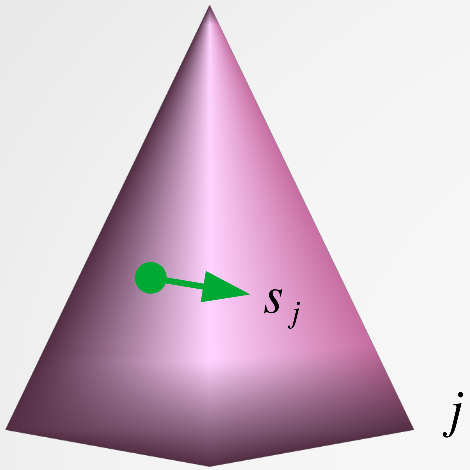
$$u_i^{n+1} = u_i^n - \frac{\Delta t}{S_i} \sum_{T \in \cup \Delta_i} \frac{1}{3} \Phi^T(u^{n+\frac{1}{2}})$$

RK2

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t/2}{\sum_{T \in \cup \Delta_i} \beta_i^T S_T} \sum_{T \in \cup \Delta_i} \beta_i^T \Phi^T(u^n)$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\sum_{T \in \cup \Delta_i} \beta_i^T S_T} \sum_{T \in \cup \Delta_i} \beta_i^T \Phi^T(u^{n+\frac{1}{2}})$$

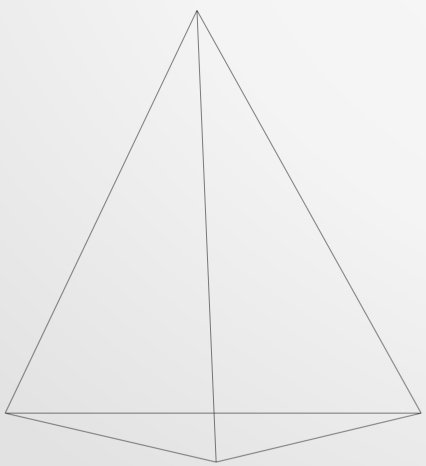
RD in Three Dimension



The flux residual is

$$\Phi^T = \sum_{j \in T} k_j u_j = \sum_{j \in T} \left(\frac{1}{3} \lambda \cdot s_j\right) u_j$$

s_j is the inward scaled normal, with magnitude equal to the plane area.

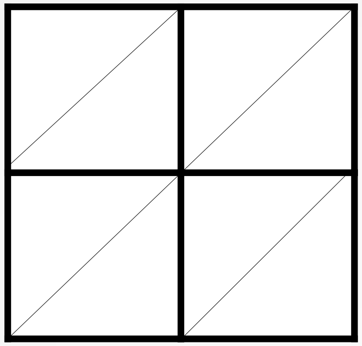


tetrahedron

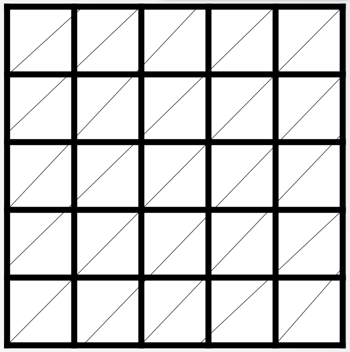
Mesh Generation

Right-running grid (RR grid)

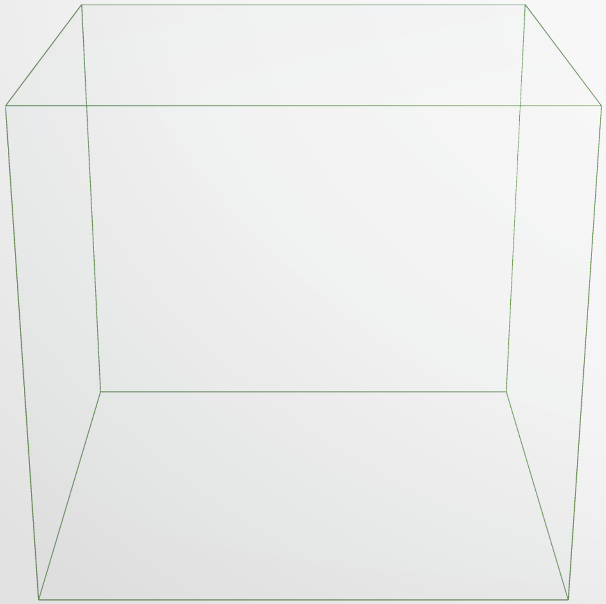
Mesh in 2D

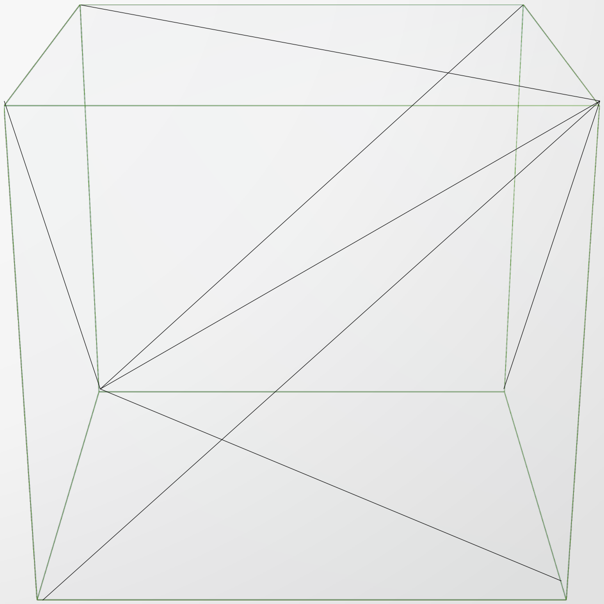


Mesh refinement



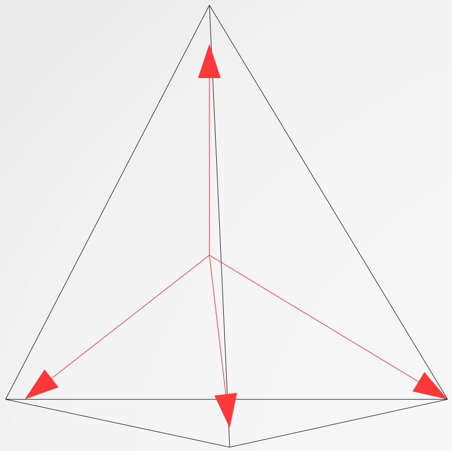
Meshing in 3D





Distribution Coefficient & Temporal Update

$$\beta_j = \frac{1}{4}$$



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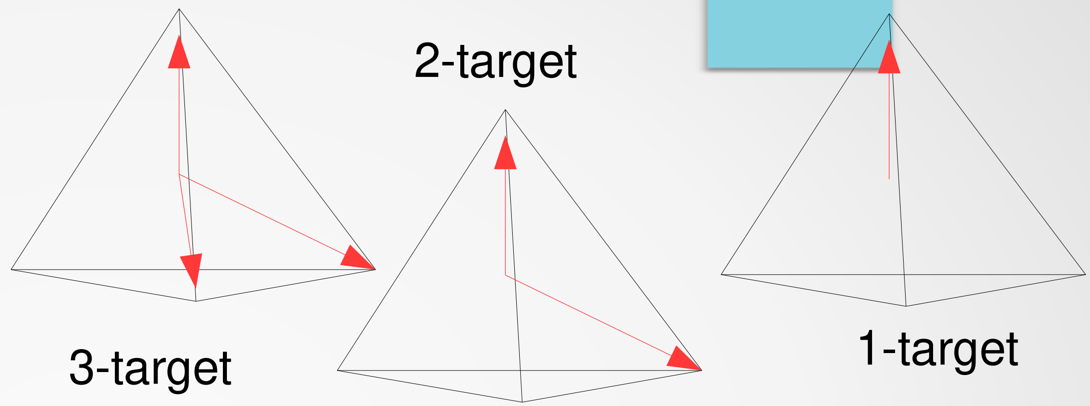


Upwind LDA

$$\beta_j = \frac{k_j^+}{\sum_{j \in T} k_j^+}$$

RK2

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t/2}{V_i} \sum_{T \in \cup \Delta_i} \frac{1}{4} \Phi^T(u^n)$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{V_i} \sum_{T \in \cup \Delta_i} \frac{1}{4} \Phi^T(u^{n+\frac{1}{2}})$$

RK2

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t/2}{\sum_{T \in \cup \Delta_i} \beta_i^T V_T} \sum_{T \in \cup \Delta_i} \beta_i^T \Phi^T(u^n)$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\sum_{T \in \cup \Delta_i} \beta_i^T V_T} \sum_{T \in \cup \Delta_i} \beta_i^T \Phi^T(u^{n+\frac{1}{2}})$$

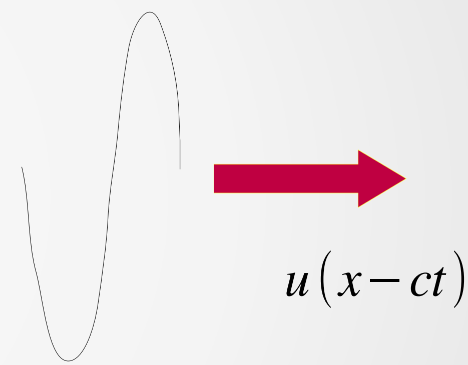
Possible Test Cases?

The solution must fulfill the governing equation

$$\frac{\partial u}{\partial t} + \lambda \cdot \nabla u = 0$$

One example that we can think of is

$$u(x, y, z, t) = u_0 \cos(k_x x + k_y y + k_z z - \omega t)$$



provided that

$$\lambda_x k_x + \lambda_y k_y + \lambda_z k_z = \omega$$