Residual Distribution Schemes for Parabolic Equations

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Overview

Introduction

- Focus is to incorporate viscous terms in the residual-distribution scheme
- Investigated in the context of 2-dimensional advection-diffusion problems

$$u_t + au_x + bu_y = \nu(u_{xx} + u_{yy}) \tag{1}$$

- The semi-discrete form of the governing equation for node i is of the form:

$$S_i \frac{du_i}{dt} + \sum_{T \in V_i} (\phi^T_{i,inv} + \phi^T_{i,vis}) = 0 \tag{2}$$

- There are different approaches to evaluate the viscous component of the residual for each cell

Viscous Component of Fluctuation Splitting

- Begin with usual residual distribution approach for a triangle

$$\phi^T_{vis} = -\iint_T \nu(u_{xx} + u_{yy})\, dx\, dy \tag{3}$$

- Using Gauss's theorem,

$$\phi^T_{vis} = -\oint_{\partial T} \nu \nabla u.\, d\mathbf{n} \tag{4}$$

- Since the solution is assumed to vary linearly over each triangle, the gradient for a triangle is constant
- This results in $\phi^T_{vis} = 0$ for the triangle

Viscous Component of Fluctuation Splitting

- Therefore we need a higher-order reconstruction and extension of the computational stencil are required to recover the ∇u on the boundary
- There are generally 2 alternatives
  - Nodal gradients - using Least Squares approach
  - Edge based gradients - using an arithmetic average approach with neighbouring triangle to recover the gradients at the midpoint
- However these approaches have the disadvantage of a larger stencil
- Another approach will be considered to keep the stencil compact

Viscous Component of Fluctuation Splitting

- Consider the median dual cell S_i

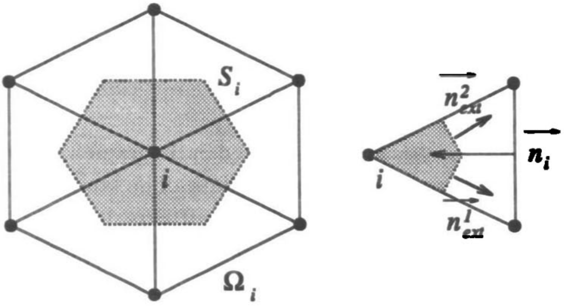

Figure: Median dual cell around node i

Viscous Component of Fluctuation Splitting

- Solve,

$$\iint_{S_i} \nu(u_{xx} + u_{yy})\, dx\, dy = \oint_{\partial S_i} \nu \nabla u.\, d\mathbf{n} \qquad (5)$$

$$\oint_{\partial S_i} \nu \nabla u.\, d\mathbf{n} = \sum_{T \in V_i} \oint_{\partial S_i \cap T} \nu \nabla u.\, d\mathbf{n} \qquad (6)$$

- Where the boundary $\partial S_i \cap T$ for each triangle T, is composed of 2 segments with outward scaled normals $\mathbf{n}^{\mathbf{1}}_{\mathbf{ext}}, \mathbf{n}^{\mathbf{2}}_{\mathbf{ext}}$ such that,

$$\mathbf{n}^{\mathbf{1}}_{\mathbf{ext}} + \mathbf{n}^{\mathbf{2}}_{\mathbf{ext}} = -\frac{1}{2}\mathbf{n}_{\mathbf{i}} \qquad (7)$$

Viscous Component of Fluctuation Splitting

- Thus,

$$\iint_{S_i} \nu(u_{xx} + u_{yy})\, dx\, dy = \sum_{T \in V_i} \frac{\nu}{2} \nabla u.\mathbf{n_i} \tag{8}$$

- Where for a triangle T,

$$\nabla u = \frac{\sum_{p=1}^{3} u_p \mathbf{n_p}}{2S_T} \tag{9}$$

- Note that the expression produced by the Galerkin method is identical
- Now the scheme is compact and a viscous component of the residual distribution for each triangle can be written equivalently as,

$$\phi_{i,vis}^{T} = \frac{1}{2}\mathbf{n_i}.\nu \sum_{p=1}^{3} \frac{u_p \mathbf{n_p}}{2S_T} \tag{10}$$

Viscous Component of Fluctuation Splitting

- viscous term also has an influence on the time-step restriction, which becomes,

$$CFL\frac{S_i}{\Delta t_i} = \sum_{T \in V_i} \frac{1}{restri^i_{inv} + restri^i_{vis}} \tag{11}$$

- Where $restri^i_{vis}$ is the diffusive time-step restriction (determined by positivity of the scheme), which is given by:

$$restri^i_{vis} = \frac{\nu \|\mathbf{n_i}\|^2}{4S_T} \tag{12}$$

Viscous Residual Solver Validation

- Validation for the viscous residual using Galerkin approach for 201 nodal points in x and y directions and Dirichlet boundary conditions for the pure diffusion problem with $\nu = 1$.
- Where

$$u_{left} = u_{bottom} = 0 \tag{13}$$

$$u_{right} = \sin(\pi y), u_{top} = -\sin(\pi x) \tag{14}$$

- Solving a pure diffusion problem for the domain $-1 \leq x \leq 0$ and $1 \leq y \leq 0$
- The analytic solution is:

$$u(x, y) = \frac{1}{\sinh \pi}[\sinh(\pi(x+1)) sin(\pi y) + \sinh(\pi y) sin(\pi(x+1))] \tag{15}$$

Grid

- Right running grid with skewness = 0.3

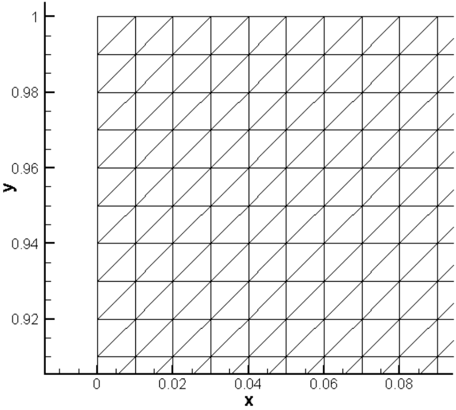

Figure: Close-up of the grid

Viscous Residual

- Numerical solution with skewness = 0.3

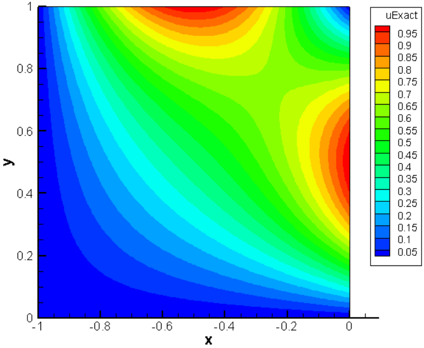

Figure: Exact solution

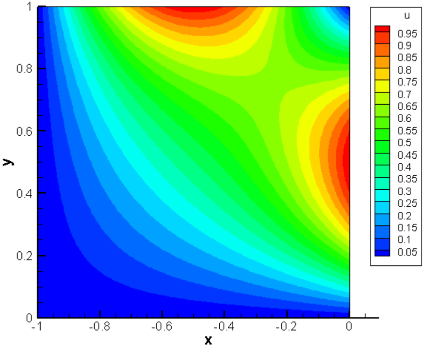

Figure: Numerical solution

Results

Comparison of the exact and numerical solution for 3 different y slices.

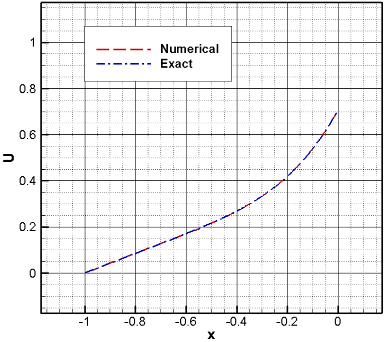

Figure: Comparison of the exact and numerical solution for y = 0.25

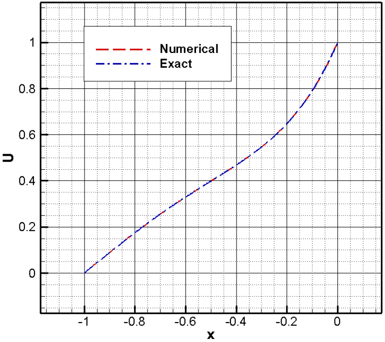

Figure: Comparison of the exact and numerical solution for y = 0.50

Results

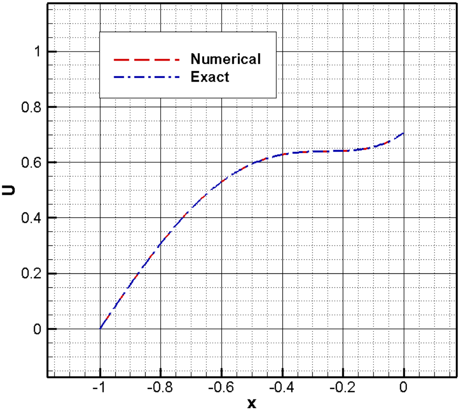

Figure: Comparison of the exact and numerical solution for y = 0.75

Results - Linear Advection-Diffusion Equation

- Solving for the domain $0 \leq x \leq 1$ and $1 \leq y \leq 0$ with $\nu = 0.1$ and $a = 7.0, b = 4.0$
- Using Dirichlet boundary conditions from the exact solution
- The exact solution is given as:

$$u(x, y) = -\cos(\pi\eta)\exp[0.5\xi(1 - \sqrt{1 + 4\pi^2\nu^2})/\nu] \tag{16}$$

- Where

$$\eta = bx - ay \tag{17}$$

$$\xi = ax + by \tag{18}$$

Results - Linear Advection-Diffusion Equation

- Inviscid residual computed using classic LDA scheme with grid skewness of 0.3

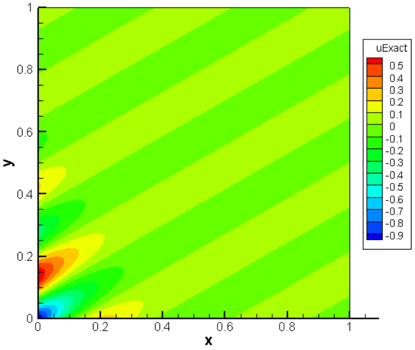

Figure: Exact solution for the linear advection-diffusion equation

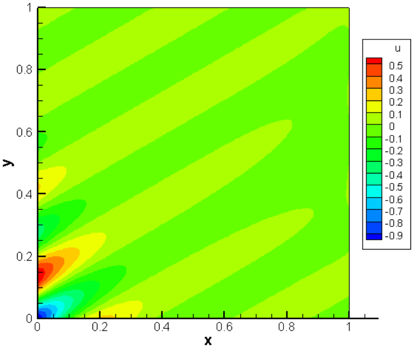

Figure: Numerical solution with 101 nodes in x and y direction

Results - Linear Advection-Diffusion Equation

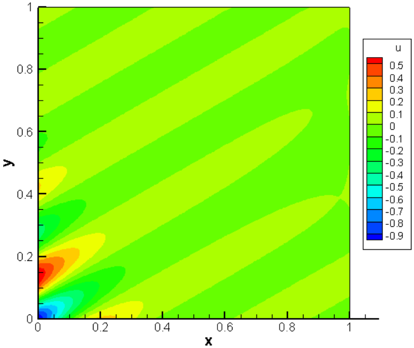

Figure: Numerical solution with 201 nodes in x and y direction

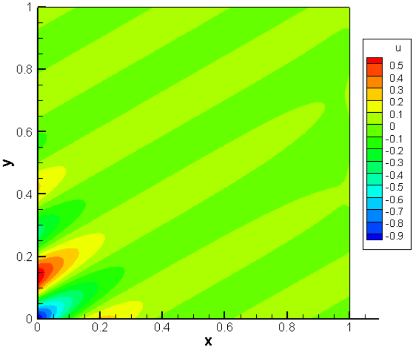

Figure: Numerical solution with 301 nodes in x and y direction

Results - Linear Advection-Diffusion Equation

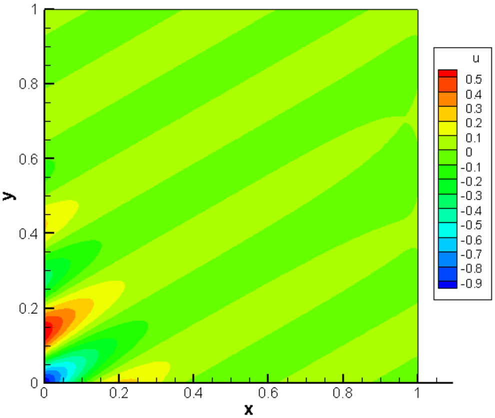

Figure: Numerical solution with 401 nodes in x and y direction

Results - Linear Advection-Diffusion Equation

Comparison of the exact and numerical solution for 3 different y slices for 201 nodes in x and y directions.

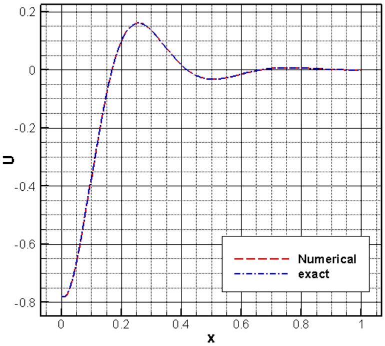

Figure: Comparison of the exact and numerical solution for y = 0.025

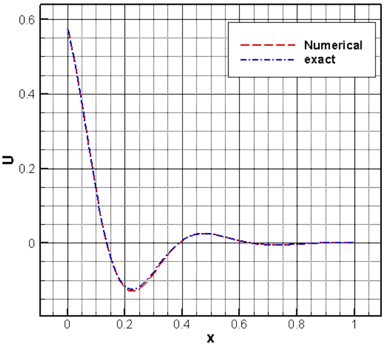

Figure: Comparison of the exact and numerical solution for y = 0.15

Results - Linear Advection-Diffusion Equation

Comparison of the exact and numerical solution for 2 different x slices.

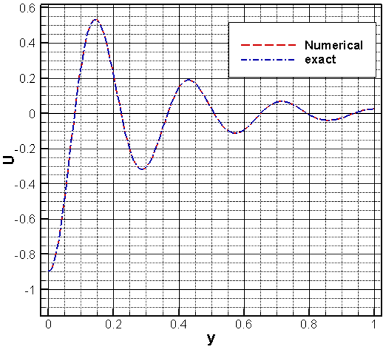

Figure: Comparison of the exact and numerical solution for x = 0.015

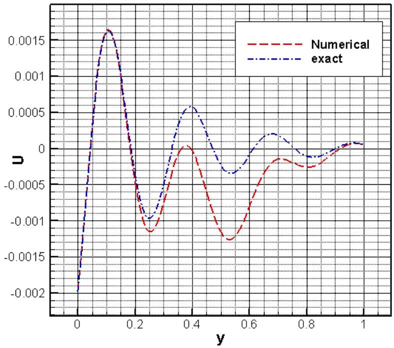

Figure: Comparison of the exact and numerical solution for x = 0.95

Results

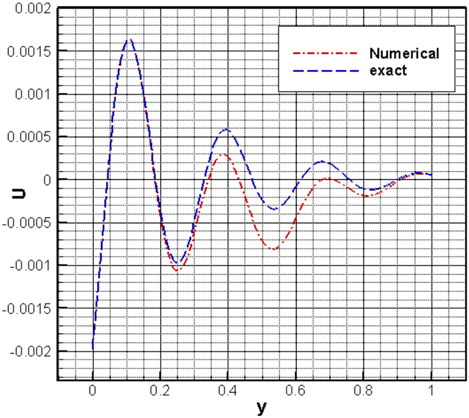

Figure: Comparison of the exact and numerical solution for y = 0.75 for 401 nodes

Order of Accuracy

Grid Spacing	L2-Norm Error	Order of Accuracy
0.010	2.19E-03	2.020323697
0.005	5.34E-04	
0.0033	2.37E-04	
0.0025	1.33E-04	

Table: Advection

Grid Spacing	L2-Norm Error	Order of Accuracy
0.010	2.04E-03	0.950064625
0.005	1.07E-03	
0.0033	7.22E-04	
0.0025	5.46E-04	

Table: Advection-Diffusion

Order of Accuracy for different skewness

Skewness	Order of Accuracy
0.3	0.950064625
0.4	0.847073513
0.5	0.744171265

Table: Order of Accuracy for Advection-Diffusion

L2-Norm Error Comparison

Grid Spacing	Classic LDA	Weighted K Approach LDA
101	2.04E-03	1.17E-03
201	1.07E-03	6.17E-04

Table: L2-Norm Error Comparison

Thank You