Alternative Approach to Prove Linear Preserving (LP) Property of RD Schemes

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Overview

Basic Proof

- The Linear Preserving (LP) property requires the scheme to preserve the exact steady state solution when the solution varies linearly in space for arbitrary grids
- A scheme that is LP will guarantee second-order accuracy for spatial discretization at steady state
- Recall the update for node i

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{S_i} \sum_{T \in V_i} \beta_i^T \phi^T \tag{1}$$

- where ϕ^T is the total residual and β_i is the distribution coefficient

Basic Proof 2

- At steady state, is we substitute the exact solution at the nodes, we get $\phi^T = 0$
- If we assume that β_i is bounded as ϕ^T approaches zero, the from Eq. 1 we get

$$u_i^{n+1} = u_i^n \tag{2}$$

- Thus, we preserve the exact solution which proves LP
- Conversely, a change in u_i is only possible if $\beta_i^T \phi^T$ is not zero
- This implies that β_i is not bounded as ϕ^T approaches zero
- Thus, boundedness of the distribution coefficient is a sufficient condition for LP
- For example, LDA distribution coefficients defined as $\beta_i = \frac{k_i^+}{\sum k^+}$ is always between 0 and 1 (thus bounded)

Proof For Flux-Difference Approach

- Recall, the flux difference signal to node i is

$$\phi_i^{FD} = \frac{1}{2}(\vec{f_i} - \vec{f^*}) \cdot \vec{n_i} + \text{A.T} \tag{3}$$

- where A.T are the artificial terms
- Naturally, to prove LP, we would need to cast this approach in the classic RD form which is

$$\phi_i^{Flux-Diff} = \beta_i^{Flux-Diff} \phi^T \tag{4}$$

where

$$\beta_i^{Flux-Diff} = \frac{\frac{1}{2}(\vec{f_i} - \vec{f^*}) \cdot \vec{n_i} + \text{A.T}}{\phi^T} = \frac{(2ui - uj - uk)k_i + \text{A.T}}{\phi^T} \tag{5}$$

Proof LP For Flux-Difference Approach

- From Eq. 5, as ϕ^T approaches zero, $\beta_i^{Flux-Diff}$ is unbounded
- From the reasoning above, the flux-difference approach is not LP
- However, from numerical experiments we know that the flux-difference approach is 2nd order accurate and should be LP
- We first conclude that we can't interpret the flux-difference approach as done in Eq. 4
- To prove LP for flux-difference approach, we need an alternate approach

Proof LP For Flux-Difference Approach

- At steady state, the sum of signals to node i from all the triangular elements that share the node is,

$$\sum_{T, i \in T} \phi_i^T = 0. \tag{6}$$

- Introducing a smooth function $\Phi \in C^1$ and take the product with Eq. 6 as well as taking the summation over all nodes, the following relation is obtained.

$$\sum_{i} \Phi_i \cdot \sum_{T, i \in T} \phi_i^T = 0 \tag{7}$$

- Recall for a triangular element T,

$$\sum_{i \in T} \phi_i^T = \iint_T \vec{\nabla} \cdot \vec{F}^h dA. \tag{8}$$

Proof LP For Flux-Difference Approach

- Introduce for a triangular element T with vertices/nodes i, j, k a function Φ^T defined as $\Phi^T = \frac{\Phi_i + \Phi_j + \Phi_k}{3}$ and take the product with Eq. 8 which results in,

$$\sum_T \Phi^T \sum_{i \in T} \phi_i^T = \sum_T \Phi^T \iint_T \vec{\nabla} \cdot \vec{F}^h dA. \tag{9}$$

Now subtracting the terms in Eq. 7 from Eq. 9, we obtain

$$\sum_T \Phi^T \iint_T \vec{\nabla} \cdot \vec{F}^h dA + \sum_T \sum_{i \in T} (\Phi_i - \Phi^T) \cdot \phi_i^T = 0 \tag{10}$$

To obtain second-order accuracy at steady state, the second terms in Eq. 10 must be of order h^2. Which is,

$$\sum_i (\Phi_i - \Phi^T) \cdot \Big(\sum_{T, i \in T} \phi_i^T \Big) = O(h^2) \tag{11}$$

This is true when the signal ϕ_i^T is $O(h^3)$ since for a bounded domain the number of nodes ($\sum_i$) is $O(h^{-2})$ and $(\Phi_i - \Phi^T)$ is $O(h)$.

Proof LP For Flux-Difference Approach

- As an example, signal for classic LP(bounded distribution coefficients β_i) RD is of the form,

$$\phi_i^{T,LDA} = \beta_i \phi^T. \tag{12}$$

- Recall that $\phi^T = \oint \vec{F} \cdot \hat{n} dl$ and approximating the flux with trapezoidal rule which is $O(h^2)$ and taking the product with the scaled normals $(O(h))$ gives the $\phi^T = O(h^3)$
- Thus, with bounded coefficients, the signal will be $O(h^3)$.
- For the flux-difference approach, the baseline approach ignoring $\vec{f}^*$ where the residual over an element is evaluated using a trapezoidal rule and is of the form,

$$\phi^T = \frac{1}{2}\vec{f}_i \cdot \vec{n}_i + \frac{1}{2}\vec{f}_j \cdot \vec{n}_j + \frac{1}{2}\vec{f}_k \cdot \vec{n}_k. \tag{13}$$

- From Eq. 13 above, it is clear that residual, $\phi^T = O(h^3)$. The signal to the nodes will be,

$$\phi_{i,j,k} = \frac{1}{2}\vec{f}_{i,j,k} \cdot \vec{n}_{i,j,k}. \tag{14}$$

Proof LP For Flux-Difference Approach

- Since each signal is a portion of the total residual, ϕ^T, thus each signal will also be $O(h^3)$
- The basic flux-difference approach is second-order accurate
- With the choice of $\vec{f}^*$ being the arithmetic average, the overall second-order accuracy is also preserved
- This because $\vec{f}^*$ is defined as arithmetic average of the average of the fluxes along each edge which is,

$$\begin{aligned}\vec{f}^* =& \frac{1}{3}\left(\frac{1}{2}(\vec{f}_i + \vec{f}_j) + \frac{1}{2}(\vec{f}_j + \vec{f}_k) + \frac{1}{2}(\vec{f}_k + \vec{f}_i)\right) \\ =& \frac{1}{3}\left(\vec{f}_i + \vec{f}_j + \vec{f}_k\right)\end{aligned} \tag{15}$$

- From Eq. 15 above, since $\vec{f}^*$ is constructed as an arithmetic average of using trapezoidal rule (which is $0(h^2)$) along each edge, $\vec{f}^*$ will be $O(h^2)$
- As a result, $\vec{f}^*$ preserves the order of accuracy the full flux-difference approach is LP on arbitrary grids.

Proof For Artificial Signals

- The viscous signal can be written as ,

$$\phi^{T}_{i,vis} = \frac{\nu}{4A_T}\vec{n}_i \cdot \vec{n}_j(u_j - u_i) + \frac{\nu}{4A_T}\vec{n}_i \cdot \vec{n}_k(u_i - u_k). \qquad (16)$$

- Now, writing the artificial signals in the same form by setting $\beta = 0$ we obtain,

$$\phi^{\text{art}}_i = \alpha(u_j - u_i) + \gamma(u_i - u_k). \qquad (17)$$

- Comparing Eq. 16 and Eq. 17, we conclude that the artificial signals are a Galerkin type discretization since the terms multiplying $(u_j - u_i)$ (which are the dot product of the scaled normals) in Eq. 16 are of $O(h^2)$ while α, γ are of $O(h^2)$
- This similar form leads to the conclusion that since the Galerkin discretization is second-order, the artificial signal too will be second-order accurate

Thank You