Residual Distribution Schemes for Scalar Advection

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Overview

Introduction

- RD schemes are numerical methods that involves two steps
- The first is the residual calculation, followed by the distribution of the residuals to nodes where the residual drives the changes of the solution
- Consider, as an example, the two dimensional scalar advection equation

$$u_t + \vec{\nabla} \cdot \vec{F} = 0, \tag{1}$$

where,

$$\vec{F} = u\vec{\lambda} = (au)\hat{i} + (bu)\hat{j}$$

Also, $\vec{\lambda} = a\hat{i} + b\hat{j}$ is the characteristic vector where a and b are the advection speeds in x and y direction respectively

- We then proceed by diving the computational domain into a set of triangles $\{T\}$, and store the solution values at nodes which belong to the set of nodes $\{J\}$

- It is also assumed that the 2D spatial domain is triangulated with the type of triangle given in Figure 1 with inward scaled normals

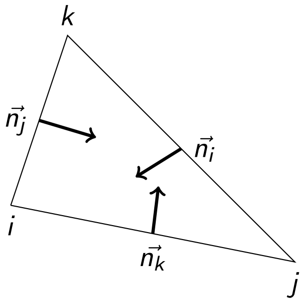


Figure: A general triangle, T (inward normals not drawn to scale).

- We define the inviscid residual ϕ_{inv}^{T} using Gauss's theorem for a triangular element T in Fig. 2 as,

$$\phi_{\text{inv}}^{T} = -\iint \vec{\nabla} \cdot \vec{F} dA = -\oint \vec{F} \cdot \hat{n} dl. \qquad (2)$$

- In a discrete form, we assume a continuous piecewise linear variation of solution and using the trapezoidal rule for quadrature, ϕ_{inv}^{T} can be expressed as,

$$\phi_{\text{inv}}^{T} \approx -\frac{1}{2}\sum_{j=1}^{3} \vec{F_j} \cdot \vec{n_j} = -\sum_{j=1}^{3} k_j u_j \qquad (3)$$

where u_j is the solution at node j and k_j is the upwind parameter defined as,

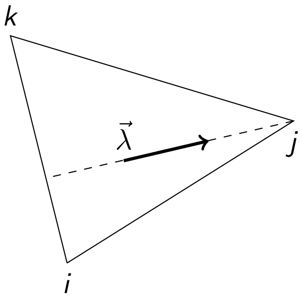
$$k_j = \frac{1}{2}\vec{\lambda} \cdot \vec{n_j} \qquad (4)$$

Introduction

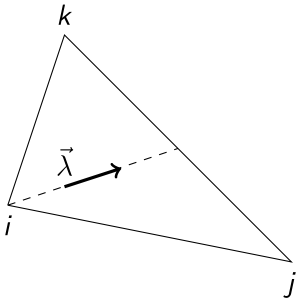
- Note that,

$$\sum_{j=1}^{3} k_j = 0 \tag{5}$$

- The sign of k_j indicates the flow direction
- For example, if $k_j > 0$ then the edge opposite node j is an inflow edge and $k_j < 0$ indicates outflow
- The definition of k_j leads to two distinct types of triangles called single-target and two-target triangles
- The single-target or Type I triangle is defined as one of the inflow parameter positive and the other two are negative.
- While the two-target or Type II triangle is when two edges have positive inflow parameters and the other is negative as shown in Figure 2



(a) Type I



(b) Type II

Figure: Two different types of triangular cells.

- After evaluating the advection residual using the vertex values, the first step is complete
- The next step is to distribute fractions of the residual or signals to the vertices of the triangle
- For consistency, the scalar distribution coefficients β_i^T sums up to unity
- Assembling the contributions from all the triangles surrounding node i, the conservative update can be written as follows,

$$S_i \frac{du_i}{dt} = -\sum_{T \in V_i} \phi_{i,\text{inv}}^T = -\sum_{T \in V_i} \beta_i^T \phi_{\text{inv}}^T \quad (6)$$

S_i is the median-dual cell area surrounding node i.

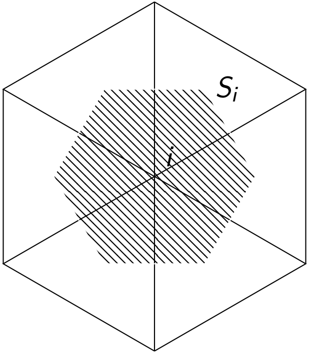


Figure: Median-Dual Area, S_i of node i

Classic RD Methods

- Looking back at Fig. 1 which distinguishes the 2 types of cells,
- The basic idea of multidimensional upwind is that we want to distribute signals to nodes with positive upwind parameter, k values
- For type Type I cells, there is only one positive k, thus the residual goes to that node
- So all classic multidimensional RD methods performs the same way for Type I cells
- The difference in classic RD methods occurs when we have Type II cells
- There are different approaches to split the residual to the nodes
- I will demonstrate 2 classical RD methods called LDA and N-scheme

LDA

- 2nd order method
- The distribution coefficient for LDA is defined as

$$\beta_i^{LDA} = \frac{k_i^+}{\sum_{p \in T} k_p^+} \tag{7}$$

- Geometrically, it is equivalent to

$$\beta_2^{LDA} = \frac{L_3}{L_2 + L_3}, \beta_3^{LDA} = \frac{L_2}{L_2 + L_3}, \tag{8}$$

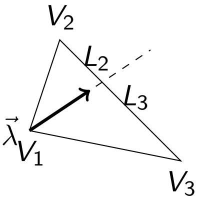


Figure: Geometric Interpretation of LDA distribution coefficients

N-scheme

- It is 1st order method and the best one so far (least diffusive 1st order)
- In the N-scheme, the advection vector is split into two components parallel to the sides opposite the downstream vertices as shown in Fig. 5

$$\vec{\lambda} = \vec{\lambda}_2 + \vec{\lambda}_3 \tag{9}$$

- In this approach, the total cell residual ϕ_τ is not determined using the main flow direction ($\vec{\lambda} = a\hat{i} + b\hat{j}$)
- The integral is determined by two new directions ($\vec{\lambda}_2 = a_2\hat{i} + b_2\hat{j}$ and $\vec{\lambda}_3 = a_3\hat{i} + b_3\hat{j}$)

N-scheme

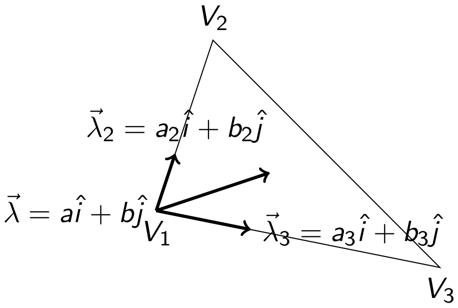


Figure: N-scheme distribution

N-scheme

- In a linear problem,it reduces to

$$\begin{aligned} \phi^{\tau} &= \iint \vec{\nabla} \cdot \vec{F} dA = \iint \vec{\lambda} \cdot \vec{\nabla} u dA = \iint \left[\vec{\lambda}_2 + \vec{\lambda}_3 \right] \cdot \vec{\nabla} u dA \\ &= \iint \vec{\lambda}_2 \cdot \vec{\nabla} u dA + \iint \vec{\lambda}_3 \cdot \vec{\nabla} u dA = \phi_2^{\tau} + \phi_3^{\tau} \end{aligned} \tag{10}$$

The signals are distributed to nodes 2 and 3 respectively, and in discrete form

$$\begin{aligned} \phi_2^{\tau} &= \sum_{edge=1}^{3} (F_x n_x + F_y n_y)_2^{edge} \Delta l^{edge}, \quad (F_x)_2 = a_2 u, \quad (F_y)_2 = b_2 u \\ \phi_3^{\tau} &= \sum_{edge=1}^{3} (F_x n_x + F_y n_y)_3^{edge} \Delta l^{edge}, \quad (F_x)_3 = a_3 u, \quad (F_y)_3 = b_3 u \end{aligned} \tag{11}$$

Other RD Methods

- Other classic RD methods include
  - Lax-Friedrich - 1st order method - The most diffusive
  - Lax-Wendroff - 2nd order method - 2 variations
  - SUPG - 2nd order method - from Finite Element approach
- A new class of RD methods known as Flux-Difference Approach developed by Prof. Dr. Farzad Ismail and Dr. Hossain Chizari

Thank You